Solutions Exercise set 2

1. We have
   \[ \frac{d}{dt} |x(t)|^2 = 2 x(t) \cdot x'(t) = 2 x(t) \cdot f(t, x(t)). \]
   Hence, the function \( r(t) = |x(t)| \) is decreasing on any time interval where \( r(t) \geq R \). It follows that \( |x(t)| \leq R_0 \) for all \( t \geq t_0 \), where \( R_0 = \max\{|x_0|, R\} \). Thus, the solution exists for all \( t \geq t_0 \) (see e.g. Corollary 2.15 in Teschl).

2. We have
   \[ \frac{d}{dt} r^2(t) = 2 x(t) \cdot f(t, x(t)) \leq 2 |x(t)| |f(t, x(t))| \leq 2 r(t) g(r(t)). \]
   It follows that
   \[ \frac{r'(t)}{g(r(t))} \leq 1, \]
   as long as \( r(t) > 0 \), since \( r(t) \) is then differentiable and \( \frac{d}{dt} r^2(t) = 2 r(t) r'(t) \). But this means that
   \[ \int_{r_0}^{r(t)} \frac{1}{g(r)} \, dr = \int_0^t \frac{r'(t)}{g(r(t))} \, dt \leq \int_0^t 1 \, dt = t. \]
   Suppose that the solution is only defined up to some time \( T_+ < \infty \) and that \( r(t) > 0 \) for all \( t \in [0, T_+) \). Then \( \lim_{t \to T_+} r(t) = \infty \) (see Corollary 2.16), giving the contradiction
   \[ \infty = \int_{r_0}^\infty \frac{1}{g(r)} \, dr \leq T_+. \]
   Hence, \( T_+ = \infty \). The condition that \( r(t) > 0 \) for all \( t \) can be avoided by just looking at the solution for \( t \) sufficiently close to \( T_+ \) (since \( r(t) \to \infty \) if \( T_+ < \infty \)).
   For the second part, simply replace \( g \) by \( g_{T_+} \).

3. Let \( x(t) \) and \( y(t) \) be two solutions with the same initial value. Then
   \[ \frac{d}{dt} (x(t) - y(t))^2 = 2(x(t) - y(t)) \cdot (f(t, x(t)) - f(t, y(t))). \]
   Assume that \( x(t_*) \neq y(t_*) \) for some \( t_* > t_0 \), and assume without loss of generality that \( x(t_*) > y(t_*) \). Then there exists some interval \( (t_1, t_*) \), with \( t_1 \geq t_0 \), such that \( x(t) > y(t) \) on \( (t_1, t_*) \)
and \(x(t_1) = y(t_1)\). But then \(x(t) - y(t) > 0\) and \(f(t, x(t)) - f(t, y(t)) \leq 0\) on the interval \((t_1, t_\star)\). Hence,

\[
\frac{d}{dt}(x(t) - y(t))^2 \leq 0, \quad t \in (t_1, t_\star).
\]

Since \(x(t_1) = y(t_1)\) this gives the contradiction \(x(t) = y(t)\) on \((t_1, t_\star)\).

4.

a) \(f(t, x)\) is a composition of continuous functions and therefore continuous. To see that \(f(t, x)\) is not loc. Lipschitz in \(x\), simply note that \(|f(t, x) - f(t, 0)|/|x| = 2\sqrt{x}/x = 2/\sqrt{x} \to \infty\) as \(x \to 0^+\).

b) We have \(x_n(t) = 0\) if \(n\) is even and \(x_n(t) = t^2\) if \(n\) is odd. Therefore, the only possible limits of subsequences of \(\{x_n\}\) are \(y_1(t) = 0\) and \(y_2(t) = t^2\). Since \(f(t, y_1(t)) = f(t, 0) = 2t \neq y_1'(t)\), we see that \(y_1(t)\) is not a solution. Since \(f(t, y_2(t)) = f(t, t^2) = 0 \neq y_2'(t)\), we see that \(y_2(t)\) is not a solution.

c) We have

\[
f(t, x) = \begin{cases} 
2t - 2\sqrt{x}, & x \geq 0, \\
2t, & x \leq 0.
\end{cases}
\]

It’s clear from this formula that \(f(t, x)\) is decreasing in \(x\).

d) If we take \(\beta = 2\), then all terms in the equation will be linear in \(t\). Assuming that \(\alpha = a^2 \geq 0\), \(a \geq 0\), and substituting \(x(t) = \alpha t^2\) into the equation, we obtain the condition

\[
a^2 = 1 - a
\]

with solution

\[
a = \frac{\sqrt{5} - 1}{2}.
\]

This gives

\[
\alpha = a^2 = \frac{3 - \sqrt{5}}{2}
\]

and

\[
x(t) = \frac{3 - \sqrt{5}}{2} t^2.
\]

5. a)
6. **a)** is asymptotically stable (the only eigenvalue is $-2$). **b)** is stable, but not asymptotically stable (the eigenvalues are $-3, \pm 2i$). **c)** is unstable (eigenvalues $-5$ and $0$, $0$ is geom. simple but alg. double). **d)** is unstable (eigenvalues $-2, 1, 3$).

7. By the hypotheses, there exist numbers $C, \alpha > 0$ such that $\|e^{tA}\| \leq Ce^{-t\alpha}$ (Corollary 3.6). By Duhamel's formula, we have

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}g(s)\,ds.$$ 

Let $\varepsilon > 0$. Since $\lim_{t \to \infty} |g(t)| = 0$, we can choose $T$ so that $|g(t)| < \varepsilon$ when $t > T$. It follows
that

\[ |x(t)| \leq |e^{tA}x_0| + \int_0^T |e^{(t-s)A}g(s)| \, ds + \int_T^t |e^{(t-s)A}g(s)| \, ds \]

\[ \leq Ce^{-\alpha t}|x_0| + \left( \max_{0 \leq s \leq T} |g(s)| \right) C \int_0^T e^{-\alpha(t-s)} \, ds + C \varepsilon \int_T^t e^{-\alpha(t-s)} \, ds \]

\[ \leq Ce^{-\alpha t}|x_0| + \left( \max_{0 \leq s \leq T} |g(s)| \right) C \frac{e^{-\alpha(t-T)} - e^{-at}}{\alpha} + C \varepsilon \frac{1 - e^{-\alpha(t-T)}}{\alpha}, \]

for \( t > T \). Choosing \( t \) sufficiently big, we can make the whole sum less than \( 2C \varepsilon/\alpha \).

If \( \lim_{t \to \infty} g(t) = g_0 \), we can set \( h(t) = g(t) - g_0 \) and \( y(t) = x(t) + A^{-1}g_0 \). We then obtain

\[ y'(t) = Ay(t) + h(t), \]

with \( \lim_{t \to \infty} h(t) = 0 \). By the preceding argument, we know that \( \lim_{t \to \infty} y(t) = 0 \). It follows that \( \lim_{t \to \infty} x(t) = -A^{-1}g_0 \).