Solutions Exercise set 3

1. The solution of $x' = x^2$, $x(0) = x_0$ is given by

$$x(t) = \frac{x_0}{1 - x_0t}.$$  

Hence,

$$\Phi(t, x) = \frac{x}{1 - xt},$$

$$I_x = \begin{cases} (-\infty, x^{-1}), & x > 0, \\ (-\infty, \infty), & x = 0, \\ (x^{-1}, \infty), & x < 0. \end{cases}$$

Note that

$$\Phi(t, \Phi(s, x)) = \frac{\frac{x}{1 - xs} \frac{x}{1 - xt}}{1 - xt} = \frac{x}{1 - sx - xt} = \frac{x}{1 - (s + t)x} = \Phi(t + s, x),$$

and $\Phi(0, x) = x$, showing directly that $\Phi$ defines a flow (see (6.11)).

2. $\gamma(x) = (0, 1)$, $\omega_-(x) = 0$ and $\omega_+(x) = 1$ if $0 < x < 1$. $\gamma(1) = \{1\}$ and $\omega_\pm(1) = \{1\}$, while $\gamma(0) = \{0\}$ and $\omega_\pm(0) = \{0\}$.

3. a) We have $\frac{dy}{dx} = \frac{x^2}{y}$, showing that the orbits are contained in the level curves $y^2 = \int x e^x \, dx = (x - 1)e^x + C$. We also have that $x'$ is positive in the upper half-plane and negative in the lower half-plane, while $y'$ is positive in the right half-plane and negative in the left half-plane. The origin is the only fixed point. Using this information, we obtain the following phase portrait.

b) In polar coordinates, $x = r \cos \theta$, $y = r \sin \theta$, the equation takes the form $r' = -\frac{1}{r}$, $\theta' = -1$. Hence there are no fixed points, and the trajectories are spirals converging towards the origin as $t \to \infty$.  

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4. First note that the only fixed points in the closed first quadrant are \((0, 0), (b, 0)\) and \((0, d)\).

Note that \(x' = 0\) on the \(y\)-axis. Hence, any solution starting on the \(y\)-axis remains there. Since different orbits can’t cross, no solution starting in the first quadrant can leave through the \(y\)-axis. Similarly, we find that the \(y' = 0\) on the \(x\)-axis, so that no solution starting in the first quadrant can leave through the \(x\)-axis. It follows that the open first quadrant is an invariant set.

Let \(A = \{(x, y) : x > 0, y > 0, x + y < d\}\), \(B = \{(x, y) : x > 0, y > 0, d < x + y < b\}\) and \(C = \{(x, y) : x > 0, y > 0, x + y > b\}\). For \((x, y) \in C\) we have that \(x' < 0\) and \(y' < 0\). Any solution starting in \(C\) will therefore either exit into \(B\) or remain in \(C\) for all future times. Note that by monotonicity, the solution will be contained in the compact set \(\{(x, y) : x \geq 0, y \geq 0, x + y \leq x_0 + y_0\}\) and therefore it is complete. If it remains in \(C\) for all \(t \geq 0\), it will have a limit on \(\partial C\) as \(t \to \infty\) by monotonicity. The limit has to be a fixed point (by the invariance of limit sets) and hence it must be \((b, 0)\).

For \((x, y) \in B\) we have that \(x' > 0\) and \(y' < 0\). On the boundary to \(C\) and \(A\) the vector field points into \(B\). Hence, a solution starting in \(B\) must remain in \(B\) for all future times and it must therefore converge to a fixed point on \(\partial B\). There are two such possible fixed points: \((b, 0)\) and \((0, d)\). From the sign of \(x'\) and \(y'\) it follows that the only possible limit is \((b, 0)\).

Finally, for \((x, y) \in A\), we have that \(x' > 0\) and \(y' > 0\). Any solution starting in \(A\) must eventually leave into \(B\) (it can’t converge to any of the fixed points on \(\partial A\)).

5. Note that \(r(t) \to 1\) as \(t \to \infty\) as long as \(r(0) \neq 0\) (this follows from the fact that \(\dot{r} = r(1 - r^2) > 0\) when \(0 < r < 1\) and \(\dot{r} < 0\) when \(r > 1\)). Similarly, \(\theta(t) \to 2\pi\) for every \(\theta(0) \in (0, 2\pi)\) since \(\dot{\theta} = 2\sin(\theta/2)^2 > 0\) when \(\theta \in (0, 2\pi)\). This means that every solution not starting at the origin converges to the point \((1, 0)\) as \(t \to \infty\).
On the other hand, the orbit of any point on the unit circle with \( \theta \in (0, 2\pi) \) is the unit circle minus the point \((1, 0)\). Thus, we can find points arbitrarily close to \((1, 0)\) on the unit circle, which will leave any given, sufficiently small neighbourhood of the origin. If we choose \(U(x_0)\) in the definition of stability as any set which doesn’t contain the whole unit circle, then, no matter how we choose the neighbourhood \(V(x_0) \subseteq U(x_0)\), there exist solutions starting in \(V(x_0)\) which eventually leave \(U(x_0)\).

6. For example
   a) \(L(x,y) = x^2 + y^2\),
   b) \(L(x,y) = x^2 + 2y^2\),
   c) \(L(x,y) = x^6 + 3y^2\).

7. Set \(L(x,y) = x^2 + y^2\). Then \(\dot{L}(x,y) = 2x(-x^3 + y) + 2y(-x) = -2x^4 \leq 0\). Thus, any orbit on which \(L\) is constant must be contained in the \(y\)-axis. But there we have \(x' = y \neq 0\) if \(y \neq 0\). Hence, the only such orbit is the set \(\{0, 0\}\). It follows that the origin is asymptotically stable by Krasovskii-LaSalle. Furthermore, any point in the plane is contained in some set \(S_\delta = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq \delta\}\) and \(S_\delta\) is compact. Hence, any solution converges to the origin as \(t \to \infty\).

8. The statement is actually incorrect. There can also be global solutions which converge to some fixed points \((x,y) = (x_\pm,0)\) as \(t \to \pm \infty\) (possibly with \(x_- = x_+\)). Any solution will be contained in the level set \(\frac{y^2}{x^2} + U(x) = E_0\) for some \(E_0\). Since \(\lim_{|x| \to \infty} U(x) = \infty\), the level set is compact and hence the solution is global. Suppose that \(U(x_0) < E_0\) and let \((x_1, x_2)\) be the maximal (bounded) open interval on which \(U(x) < E_0\). Such an interval exists because \(\lim_{|x| \to \infty} U(x) = \infty\).

   Suppose that \(U'(x_1) \neq 0\) and \(U'(x_2) \neq 0\). Since the level curve is symmetric about the \(x\)-axis, it then forms a closed loop without any fixed points. Hence the solution is periodic.

   If \(U'(x_2) = 0\) while \(U'(x_1) \neq 0\), say, then the solution will converge to \((x_2, 0)\) as \(t \to \pm \infty\).

   If \(U'(x_1) = U'(x_2) = 0\), then the solution will converge to \((x_1, 0)\) as \(t \to -\infty\) and \((x_2, 0)\) as \(t \to \infty\) or vice versa, depending on whether \(y_0\) is positive or negative.

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If \( U(x_0) = E_0 \), then \( y_0 = 0 \). It could happen that the level curve just contains the point \((x_0, 0)\), in which case it is a fixed point. Otherwise, \((x_0, 0)\) is the endpoint of an interval \((x_1, x_2)\) such as above, and we can apply the same argument.

9. The energy is no longer conserved. We have

\[
\frac{d}{dt} \left( \frac{y^2}{2} + 1 - \cos x \right) = -\eta y^2 \leq 0,
\]

with equality only if \( y = 0 \). On the other hand, if \( y = 0 \) then \( y' = -\sin(x) \neq 0 \) if \( x \neq \pi n \), \( n \in \mathbb{Z} \). Thus, except for the fixed points there are no orbits completely contained in a set of constant energy. The origin is therefore asymptotically stable by Krasovskii-LaSalle, and any solution starting in the set \( \{(x, y) \in \mathbb{R}^2 : y^2/2 + 1 - \cos x < 2\} \) converges to the origin. Below is a sketch of the phase portraits for \( \eta = 0 \) and \( \eta = 0.5 \).