Solutions Exercise set 4

1. a) The fixed points are solutions of

\[
\begin{cases}
3 - 2y - xy - x^2 - y^2 = 0 \\
2x + x^2 = 0.
\end{cases}
\]

The latter equation factorizes into \(x(2 + x) = 0\), so that \(x = 0\) or \(x = 2\). Substitution of \(x = 0\) into the first equation results in

\[
3 - 2y - y^2 = 0 \iff y = -3 \text{ or } y = 1.
\]

When \(x = 2\) we instead obtain

\[-1 - y^2 = 0,
\]

which has no real solutions. The fixed points are \((0, -3)\) and \((0, 1)\).

The Jacobian matrix of \(f(x, y) = (3 - 2y - xy - x^2 - y^2, 2x + x^2)\) is

\[
f'(x, y) = \begin{pmatrix}
-y - 2x & -2 - x - 2y \\
2 + 2x & 0
\end{pmatrix}.
\]

\((0, -3)\)

The matrix for the linearization at \((0, -3)\) is

\[
f'(0, -3) = \begin{pmatrix}
3 & 4 \\
2 & 0
\end{pmatrix},
\]

with eigenvalues \(\frac{3}{2} \pm \frac{\sqrt{41}}{2}\). This fixed point is therefore unstable.

\((0, 1)\)

The matrix for the linearization at \((0, 1)\) is

\[
f'(0, 1) = \begin{pmatrix}
-1 & -4 \\
2 & 0
\end{pmatrix},
\]

with eigenvalues \(-\frac{1}{2} \pm i\frac{\sqrt{41}}{2}\). The fixed point is asymptotically stable.

b) The fixed points \((\pm 1, 0)\) are asymptotically stable. \((0, 0)\) is unstable.

c) The fixed points \((0, n\pi), n \in \mathbb{Z}\), is unstable for even \(n\) and asymptotically stable for odd \(n\).

d) The fixed point \((0, 0, 0)\) is unstable.

2. a) \[
\begin{cases}
x' = y, \\
y' = x^2 + y - 1.
\end{cases}
\]

The fixed points are \((\pm 1, 0)\). Both are unstable.
b) \[
\begin{align*}
    x' &= y, \\
    y' &= \frac{e^x - 1}{y+1}
\end{align*}
\]

The only fixed point is \((0, 0)\) and it’s unstable.

3. The origin is asymptotically stable if \(0 < a < 1\). It’s unstable if \(a < 0\) or \(a > 1\). If \(a = 0\) or \(a = 1\), the linearization can’t be used to determine the stability of the origin.

4. We have \[
\frac{d}{dt} \log x(t) = \frac{x'(t)}{x(t)} = 1 - y(t).
\]

Integrating over one period, we find that

\[
0 = \log x(T) - \log x(0) = \int_0^T (1 - y(t)) \, dt = T - \int_0^T y(t) \, dt.
\]

Hence,

\[
\frac{1}{T} \int_0^T y(t) \, dt = 1.
\]

A similar computation involving \(\log y(t)\) shows that

\[
\frac{1}{T} \int_0^T x(t) \, dt = 1.
\]

5. Follow the instructions. Note that it should say \(x(t) > x_1\) instead of \(x(t) < x_1\).

6. The first quadrant is invariant (the axes are filled with orbits). The origin is an unstable fixed point and so is \((C/D, A/B)\).

The orbits (first quadrant) are implicitly given by

\[
A \ln y - By = C \ln x - Dx + E,
\]

\(E\) constant.

Below is a sketch of the phase portrait for \(A = B = C = D = 1\).
7. Write the equation as the first order system
\[
\begin{align*}
    x'_1 &= x_2, \\
    x'_2 &= (4 - x_1^2 - x_2^2)x_1,
\end{align*}
\]
where \( x_1 = y, \ x_2 = y' \). Introducing polar coordinates, we obtain the system
\[
\begin{align*}
    r' &= r(5 - r^2) \cos \theta \sin \theta, \\
    \theta' &= (4 - r^2) \cos^2 \theta - \sin^2 \theta.
\end{align*}
\]
Taking \( r = \sqrt{5} \), we obtain
\[
\begin{align*}
    r' &= 0, \\
    \theta' &= -1.
\end{align*}
\]
Thus, \( r(t) = \sqrt{5}, \ \theta(t) = -t \) is a solution. Going back to the original variables, we obtain the periodic, non-constant solution \( y(t) = \sqrt{5} \cos(t) \) (or more generally, \( y(t) = \sqrt{5} \cos(t - t_0) \), where \( t_0 \in \mathbb{R} \) is arbitrary).

8. The fixed points satisfy the system of equations
\[
\begin{align*}
    x + y - x(x^2 + 2y^2) &= 0, \\
    -x + y - y(x^2 + 2y^2) &= 0.
\end{align*}
\]
Multiplying the first equation by \( x \), the second by \( y \) and adding the two equations gives
\[
(x^2 + y^2)(1 - x^2 - 2y^2) = 0.
\]
If \((x, y) \neq (0, 0)\), we obtain that \( x^2 + 2y^2 = 1 \), in which case (1) reduce to
\[
\begin{align*}
    -x &= 0, \\
    y &= 0.
\end{align*}
\]
But this also gives \((x, y) = (0, 0)\). Hence, the origin is the only fixed point.
Setting \( r(t) = \sqrt{x(t)^2 + y(t)^2} \), we find that
\[
\frac{dr^2}{dt} = 2x x' + 2yy' \\
   = 2x(x + y - x(x^2 + 2y^2)) + 2y(-x + y(x^2 + 2y^2)) \\
   = 2(x^2 + y^2)(1 - x^2 - 2y^2).
\]
On any circle \( x^2 + y^2 = r_1^2 \) with \( 0 < r_1 \leq 1/\sqrt{2} \) we have that
\[
x^2 + 2y^2 \leq 2(x^2 + y^2) = 2r_1^2 \leq 1.
\]
Hence, the radius of the solution is increasing as long as \( r(t) \leq 1/\sqrt{2} \). On any circle with radius \( r_2 \geq 1 \), we have on the other hand that
\[
x^2 + 2y^2 \geq x^2 + y^2 = r_2^2 \geq 1.
\]
Hence the radius is decreasing as long as \( r(t) \geq 1 \). It follows that the compact set
\[
K = \{(x, y) \in \mathbb{R}^2: \frac{1}{2} \leq x^2 + y^2 \leq 1\}
\]
is a positively invariant, compact set, which doesn't contain any fixed points. By the Poincaré-Bendixson theorem (Lemma 7.13) it therefore contains a regular periodic orbit, corresponding to a non-constant periodic solution. In fact, the \( \omega_+ \) limit set of any point in \( K \) will be such an orbit.