Solutions Exercise set 7

1. a) Eigenvalues \( \lambda_n = \frac{(2n+1)^2 \pi^2}{4} \). Normalized eigenfunctions \( u_n(x) = \sqrt{2} \cos \left( \frac{(2n+1)\pi x}{2} \right) \).

b) 
\[
1 = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4}{(2n+1)\pi} \cos \left( \frac{(2n+1)\pi x}{2} \right).
\]

c) 
\[
x^2 - 1 = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cdot 32}{(2n+1)^3\pi^3} \cos \left( \frac{(2n+1)\pi x}{2} \right).
\]

d) The convergence is uniform for \( g \) but not for \( f \). \( g \) is in the domain of the operator \( (g \in C^2, g'(0) = 0, g(1) = 0) \), but not \( f \) (since \( f(1) \neq 0 \)). Note that the series for \( f \) doesn't even converge pointwise to \( f \) at \( x = 1 \) (since the partial sums \( s_N \) satisfy \( s_N(1) = 0 \)).

2. We have 
\[
\int_0^1 |g(x)|^2 \, dx = \sum_{n=0}^{\infty} |c_n|^2,
\]
where 
\[
c_n = (-1)^{n+1} \frac{16\sqrt{2}}{(2n+1)^3\pi^3}.
\]
Hence, 
\[
\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{512} \int_0^1 (x^2 - 1)^2 \, dx = \frac{\pi^6}{960}.
\]

3. We have \( Lf_n = n^2f_n \) and thus 
\[
\|L f_n\| = n^2 \|f_n\|.
\]
It follows that 
\[
\|L(\|f_n\|-^{-1} f_n)\| = \frac{\|L f_n\|}{\|f_n\|} = n^2 \to \infty
\]
as \( n \to \infty \). But this means that 
\[
\|L\| = \sup_{\|f\|=1} \|L f\| = \infty,
\]
so that \( L \) is unbounded.
4. By the Lagrange identity (5.57), we have that
\[ \langle g, Lf \rangle - \langle Lf, f \rangle = \hat{W}_a(g, f) - \hat{W}_b(g, f), \]
where
\[ \hat{W}_a(g, f) = p(x)(g(x)f'(x) - g'(x)f(x)). \]
If \( f, g \in D(L) \), then
\[ \hat{W}_a(g, f) - \hat{W}_b(g, f) = \overline{\langle a \rangle}p(a)f'(a) - p(a)\overline{\langle a \rangle}f(a) - \overline{\langle b \rangle}p(b)f'(b) + p(b)\overline{\langle b \rangle}f(b) \]
\[ = \overline{\langle b \rangle}p(b)f'(b) - p(b)\overline{\langle b \rangle}f(b) - \overline{\langle b \rangle}p(b)f'(b) + p(b)\overline{\langle b \rangle}f(b) \]
\[ = 0, \]
so that \( \langle g, Lf \rangle = \langle Lg, f \rangle \).
The boundary conditions are called periodic, since if \( p, q \) and \( r \) are periodic functions on \( \mathbb{R} \), then these are exactly the boundary conditions that a periodic solution of the equation on \( \mathbb{R} \) would satisfy.
The eigenvalues are \( \lambda_n = n^2 \), \( n = 0, 1, 2, \ldots \). The eigenvalue 0 is simple with eigenfunction \( u_0(x) \equiv 1 \). The other eigenvalues are double with eigenvectors \( u_n(x) = \cos(nx) \) and \( v_n(x) = \sin(nx) \). This is in contrast to the case of separated boundary conditions, in which case the eigenvalues are always simple.

5. If \( \lambda \neq 0 \) is an eigenvalue and \( u \) a corresponding eigenvector, then \( A(\lambda^{-1}u) = u \), so \( u \in \operatorname{Ran}(A) \). Hence,
\[ u = \sum_{j=0}^{\infty} \langle u_j, u \rangle u_j. \]
If \( \lambda \neq \alpha_j \) for all \( j \), then \( \langle u_j, u \rangle = 0 \) for all \( j \) (eigenvectors corresponding to different eigenvalues are orthogonal). Hence we obtain the contradiction \( u = 0 \). Thus \( \lambda = \alpha_n \) for some \( n \). Since \( \lim_{j \to \infty} \alpha_j = 0 \), there are only finitely many \( j \) for which \( \alpha_j = \alpha_n \). Thus
\[ u = \sum_{j: \alpha_j = \alpha_n} \langle u_j, u \rangle u_j, \]
where the sum is finite.
If \( \operatorname{Ran}(A) \) is dense, the eigenfunction expansion holds for any \( u \in H \). If \( Au = 0 \), we have that \( \langle u_j, u \rangle = 0 \) for all \( j \) since \( \alpha_j \neq 0 \). But then \( u = 0 \), so 0 is not an eigenvalue.
Alternatively, we can use the following argument which doesn't rely on the eigenfunction expansion. Let \( \{v_n\} \) be a sequence in \( H \) with \( Av_n \to u \) as \( n \to \infty \). Then
\[ \|u\|^2 = \langle u, u \rangle = \lim_{n \to \infty} \langle u, Av_n \rangle = \lim_{n \to \infty} \langle Au, v_n \rangle = 0. \]
Hence, \( u = 0 \) and 0 is not an eigenvalue.

6. Suppose that \( u_0 \) is a solution of \(-u'' + qu = 0\) with two zeros \( c < d \). Set \( q_0(x) = q(x) \) and \( q_1(x) \equiv 0 \), so that \( q_1(x) < q_0(x) \) for all \( x \). By Sturm’s comparison theorem, any solution of the equation \( u'' = 0 \) then has a zero between \( c \) and \( d \). But this gives a contradiction since \( u_1(x) \equiv 1 \) is a solution with no zeros.

7. We use Sturm’s comparison theorem. Write the problem as \(-y'' + q_1(x)y = 0\), with \( q_1(x) = -e^x \). Then \( q_1(x) \leq -e^0 = -1 \) on \([0, \pi]\). Hence we can take \(-y'' - y = 0\) as our comparison equation \((q_0(x) = -1)\). The solution \( y_0(x) = \sin x \) vanishes at 0 and \( \pi \), implying that any solution \( y_1 \) of \( y'' + e^x y = 0 \) has a zero in \((0, \pi)\) since \( q_1(x) \neq q_0(x) \).
8. Let $\lambda \geq 4$ and set

$$q_1(x) = -\left(\frac{5}{1+x^2} + 4\right).$$

Then $q_1(x) < q_0(x)$ on $[\pi, 2\pi]$, where $q_0(x) \equiv -4$. The function $u_1(x) = \sin(2x)$ solves the equation $-u'' - 4u = 0$. It vanishes at $\pi$, $3\pi/2$ and $2\pi$. By Sturm’s comparison theorem, any solution of $-u'' + q_1(x)u = 0$ must therefore have a zero in $(\pi, 3\pi/2)$ and a zero in $(3\pi/2, 2\pi)$. Thus, any eigenfunction corresponding to an eigenvalue $\lambda \geq 4$ has at least two zeros in the interval $(\pi, 2\pi)$. If we list the eigenvalues in increasing order, $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$, then Thm 5.17 implies that $u_n$ (corresponding eigenfunction) has precisely $n$ zeros in $(\pi, 2\pi)$. But this means that $\lambda_0$ and $\lambda_1$ are both $< 4$, since otherwise, the corresponding eigenfunctions would have at least two zeros.