Solutions

1. The fixed points are given by \( x^2 - y^2 = -x^2 - xy + y^2 - x - 2y - 2 = 0 \). Thus, \( x = \pm y \) and when \( x = y \) the second equation becomes \( x^2 + 3x + 2 = 0 \), which gives \( x = -1 \) or \(-2\). When \( x = -y \), the second equation becomes \( x^2 + x - 2 = 0 \), which gives \( x = 1 \) or \(-2\). Thus, the fix points are \((-1, -1), (-2, -2), (1, -1) \) and \((-2, 2)\). The Jacobian of the right hand side is

\[
\begin{bmatrix}
2x & -2y \\
-2x - y - 1 & -x + 2y - 2
\end{bmatrix}
\]

Thus

\[
f'(-1, -1) = \begin{bmatrix}
-2 & 2 \\
2 & -3
\end{bmatrix}
\]

with eigenvalues \(- \frac{5}{2} \pm \frac{\sqrt{17}}{2}\),

\[
f'(-2, -2) = \begin{bmatrix}
-4 & 4 \\
5 & -4
\end{bmatrix}
\]

with eigenvalues \(-4 \pm 2\sqrt{5}\),

\[
f'(1, -1) = \begin{bmatrix}
2 & 2 \\
-2 & -5
\end{bmatrix}
\]

with eigenvalues \(- \frac{3}{2} \pm \frac{\sqrt{12}}{2}\) and

\[
f'(-2, 2) = \begin{bmatrix}
-4 & -4 \\
1 & 4
\end{bmatrix}
\]

with eigenvalues \(\pm \sqrt{12}\). Hence, all the fix points are unstable except \((-1, -1)\) which is asymptotically stable.

2. The problem has a unique solution for each \( f \) if and only if the homogeneous problem

\[
y''(x) + 16y(x) = 0 \\
y(0) = y(1) = 0
\]

only has the trivial solution. The general solution to the equation is \( y(x) = A \sin(4x) + B \cos(4x) \), and the boundary conditions give \( B = y(0) = 0 \) and \( A \sin(4) = y(1) = 0 \) so \( A = 0 \). Thus, the problem has a unique solution.

Please, turn over!
We try to find a Lyapunov function of the type

\[ W(x) = 4(\sin(4x) \cos(4x - 4) - \cos(4x) \sin(4x - 4)) \]

\[ = 4 \sin(4x - (4x - 4)) = 4 \sin(4) \]

The Green’s function is

\[ G(x, \xi) = \begin{cases} 
\sin(4x) \sin(4\xi - 4)/4\sin(4), & 0 \leq x \leq \xi \leq 1 \\
\sin(4\xi) \sin(4x - 4)/4\sin(4), & 0 \leq \xi \leq x \leq 1 
\end{cases} \]

Now we also have to find a solution to \( y'' + 16y = 0 \) which satisfies the boundary conditions. If \( y(x) = A \sin(4x) + B \cos(4x) \) then \( B = y(0) = 0 \) so \( A \sin(4) = 1 \). Thus the solution is

\[ y(x) = \sin(4x)/\sin(4) + \int_0^1 G(x, \xi) f(\xi) \, d\xi \]

3. We try to find a Lyapunov function of the type \( L(x, y) = ax^{2n} + by^{2m} \) with \( a, b > 0 \) and \( n, m \geq 1 \). We have

\[ \dot{L}_f(x, y) = 2nax^{2n-1}x' + 2mby^{2m-1}y' \]

\[ = 2nax^{2n-1}(-x + y^3 \cos(x)) + 2mby^{2m-1}(-x \cos(x) - x^4 y) \]

\[ = -2nax^{2n} + (2nax^{2n-1}y^3 - 2mbxy^{2m-1}) \cos(x) - 2mbx^{2m}x^4 \]

and in order to cancel the non-negative terms we need that \( 2n-1 = 1 \) and \( 2m-1 = 3 \), thus \( n = 1 \) and \( m = 2 \). If \( 2na = 2a = 2mb = 4b \), for example \( b = 1 \) and \( a = 2 \) we find that \( L(x, y) = 2x^2 + y^4 \) satisfies

\[ \dot{L}_f(x, y) = -4x^2 - 4x^4 y^4 = -4x^2(1 + x^2 y^4) \leq 0 \]

so \( L \) is a Lyapunov function. The points in this domain where \( \dot{L}_f(x, y) = 0 \) are given by \( x = 0 \), and then \( x' = y^3 \neq 0 \) if \( y \neq 0 \). Thus no orbit except the origin is contained in this set, so the Krasovskii-LaSalle Theorem gives that the origin is asymptotically stable.

4. We can write the equation as a first order system

\[ \begin{cases} 
x' = y \\
y' = -x + y \cos(x^2 + 2y^2) 
\end{cases} \]

The origin is the only fixed point, since we obtain \( y = 0 \) from the first equation which gives \( x = 0 \) from the second. We have that

\[ \frac{d}{dt}(x^2 + y^2) = 2xy + 2y(-x + y \cos(x^2 + 2y^2)) = 2y^2 \cos(x^2 + 2y^2) \]

This is non-negative if \( x^2 + 2y^2 \leq \frac{\pi}{4} \) and non-positive if \( \frac{\pi}{2} \leq x^2 + 2y^2 \leq \frac{3\pi}{2} \). Now if \( x^2 + y^2 \leq \frac{\pi}{4} \) then \( x^2 + 2y^2 \leq 2(x^2 + y^2) \leq \frac{\pi}{2} \) so \( \frac{d}{dt}(x^2 + y^2) \geq 0 \) then. If \( \frac{\pi}{2} \leq x^2 + y^2 \leq \frac{3\pi}{2} \) then similarly \( \frac{\pi}{2} \leq x^2 + 2y^2 \leq \frac{3\pi}{2} \) so \( \frac{d}{dt}(x^2 + y^2) \leq 0 \) in this domain. It follows that the domain \( \{ \frac{x}{2} \leq x^2 + y^2 \leq \frac{\pi}{2} \} \) is an invariant set without any fix points, so by the Poincaré-Bendixson Theorem there exists a non-constant periodic orbit in this set.
5. Since $f(x, y)$ changes sign at $x = 0$ and $g(x, y)$ changes sign at $y = 0$, the origin is a fix point. Take $E(x, y) = xy$, then \( \dot{E}_F(x, y) = x(f(x, y) - y) + y(x + g(x, y)) = xf(x, y) + yg(x, y) \geq 0 \) on the orbits, with equality if and only if $x = y = 0$. If the system has a non-constant periodic orbit then $E(x, y)$ is periodic on that orbit. Since $E$ is strictly increasing on all orbits outside the origin we get a contradiction.