Problem 1

Definition 1 (Weak solution) A solution of a scalar conservation law \( u_t + f(u)_x = 0 \) is called a weak solution if
\[
\int_0^\infty \int_{-\infty}^{\infty} (\phi_t u + \phi_x f(u)) dx dt = -\int_{-\infty}^{\infty} \phi(x,0) u(x,0) dx
\]
for all functions \( \phi \in C_0^1(\mathbb{R} \times \mathbb{R}) \).

Now consider the following Burger’s equation problem:
\[
u_t + uu_x = 0,
u(0,x) = \begin{cases} u_L & x \leq 0, \\ u_R & x > 0. \end{cases}
\]

Note: A problem with piecewise constant initial data with one jump is called a Riemann problem.

Prove: The following solutions are weak solutions of the above problem in case \( u_L < u_R \):
\[
u_1(x,t) = \begin{cases} u_L & x \leq st, \\ u_R & x > st. \end{cases}
\]

with \( s = (u_L + u_R)/2 \).

\[
u_2(x,t) = \begin{cases} u_L & x < uLt, \\ x/t & uLt \leq x \leq uRt, \\ uR & x > uRt. \end{cases}
\]

The second one is called a rarefaction wave.

Problem 2
Perform a von Neumann stability analysis of the upwind scheme \( u_x(x_i) \approx \frac{u_i - u_{i-1}}{\Delta x} \) combined with the explicit Euler method with fixed step sizes for the linear advection equation with \( a > 0 \).
Problem 3
Consider the Burger’s equation

\[ u_t + uu_x = 0, \quad x \in [-1, 1], t \in [0, \infty], \]

\[ u(0, x) = \begin{cases} 
1 & x \leq 0, \\
0 & x > 0,
\end{cases} \]

\[ u(t, -1) = 1. \]

a) Discretize this using the explicit Euler method with constant mesh width \( \Delta t \) in time and an upwind finite difference method with constant mesh width \( \Delta x \) in space. Specifically, use the approximation

\[ (uu')(x_i) \approx u_i(u_i - u_{i-1})/\Delta x. \]

- Program this in a language of your choice and visualize the numerical solutions for \( t = 0, t = 1 \) and \( t = 2 \) for \( \Delta x = 1/100 \) and a stable \( \Delta t \). What do you see?
- Refine the mesh and the time step. What happens?

b) Change the space discretization by instead of approximating \( (uu')(x_i) \), approximate

\[ \left( \frac{1}{2} u^2 \right)'(x_i) \]

appropriately. What do you observe?

Return: Tuesday, April 9th, in class