SOLUTIONS TO THE EXAM 170318

1. By iterated integration, the integral can be written

$$\int_0^1 dy \int_0^y \frac{x^2+y^2}{1+y^2}\,dx = \frac{4}{3}\int_0^1 \frac{y^3}{1+y^2}\,dy.$$

The substitution $u = y^2$ brings the right hand side to the form

$$\frac{2}{3}\int_0^1 \frac{u}{1+u}\,du = \frac{2}{3}\left[u - \ln(1+u)\right]_0^1 = \frac{2}{3}(1-\ln 2).$$

2. Write $u = u(x,y) = u(s,t)$. The chain rule gives

$$\begin{aligned}
u'_x &= u'_s + u'_t\\
u''_{xx} &= u''_{ss} + 2u''_{st} + u''_{tt}\\
u'_y &= e^y u'_s - e^y u'_t\\
u''_{yy} &= e^y u'_s + e^{2y}u''_{ss} - 2e^{2y}u''_{st} - e^y u'_t + e^{2y}u''_{tt}.
\end{aligned}$$

Inserting these expressions, the differential equation simplifies to $u''_{st} = 0$ which has the general solution $u(s,t) = F(s) + G(t)$ where F and G are arbitrary C^2-smooth functions. The general solution is thus

$$u(x,y) = F(x+e^y) + G(x-e^y).$$

3. Define $f(x,y) = x^y - y^x + 1$. We compute

$$f'_y = x^y \ln x - xy^{x-1}$$

so $f'_y(1,2) = -1 \neq 0$. The implicit function theorem gives that the equation $f(x,y)=0$ can be solved for y as a continuously differentiable function of x in a neighbourhood of $(1,2)$. Differentiating through with respect to x in the identity

$$x^{y(x)} - y(x)^x + 1 = 0$$

we find that

$$x^{y(x)}(y'(x)\ln x + y(x)/x) - y(x)^x(\ln y(x) + xy'(x)/y(x)) = 0$$

for all x in some neighbourhood of 1. Inserting $(x,y) = (1,2)$, we find

$$y'(1) = 2 - 2\ln 2.$$

Since $\ln 2 < 1$ and since y' is continuous, we have $y' > 0$ in some neighbourhood of $x = 1$, i.e., y is strictly increasing there.

4. Consider the difference

$$u(x,y) = x^2 + y^2 + \pi - 2 - 4\arctan(xy).$$

We must prove that $u \geq 0$ on $\mathbb{R}^2$. For this, we shall minimize u.

First note that, since $\arctan t < \pi/2$ for all $t \in \mathbb{R}$, we have $u(x,y) > r^2 - \pi - 2$ where $r = \sqrt{x^2+y^2}$. It follows that $u(x,y) \to +\infty$ as $r \to +\infty$. Hence (since u is continuous) u attains a smallest value, and this can only occur at a critical point. Thus it suffices to prove that $u \geq 0$ at each critical point.

The critical points are solutions to the system

$$2x = \frac{4y}{1+(xy)^2} \tag{1}$$

$$2y = \frac{4x}{1+(xy)^2}. \tag{2}$$

Multiplying (1) by x and (2) by y we see that $x^2 = y^2$, i.e. $y = \pm x$. Inserting this in (1) gives $x = \pm 2x/(1+x^4)$ or $x^5 + x = \pm 2x$. With "+" we get the solutions $x = 0$ and $x = \pm 1$, while "-" gives just one real solution: $x = 0$. There are thus three critical points: $(0,0)$ and $\pm(1,1)$. We finally note that $u(0,0) = \pi - 2$ and $u(1,1) = u(-1,-1) = \pi - 4\arctan 1 = 0$. As all these values are ≥ 0, the statement is proved.

5. Let $f(x,y,x) = \sqrt{x} + \sqrt{y} + \sqrt{z}$, so M is the level-surface $f = 1$. The gradient $\nabla f(P_0)$ is normal to this level surface at P_0,

$$\nabla f(P_0) = \frac{1}{2}(x_0^{-1/2}, y_0^{-1/2}, z_0^{-1/2}).$$

The equation of the tangent plane through P_0 is given by $\nabla f(P_0) \bullet (\mathbf{x} - P_0) = 0$, or

$$x_0^{-1/2}(x - x_0) + y_0^{-1/2}(y - y_0) + z_0^{-1/2}(z - z_0) = 0.$$

Using that $x_0^{1/2} + y_0^{1/2} + z_0^{1/2} = 1$, the equation simplifies to

$$x_0^{-1/2}x + y_0^{-1/2}y + z_0^{-1/2}z = 1.$$

Setting here $(x,y,z) = (a,0,0)$ we find $a = \sqrt{x_0}$ and similarly $b = \sqrt{y_0}$ and $c = \sqrt{z_0}$. Finally $a + b + c = \sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = 1$ is independent of P_0.

6. We change variables by $u = x + y + z$, $v = x + y - y$, $w = x - y - z$. The domain T corresponds to the tetrahedron

$$\tilde{T}: \; u, v, w \geq 0, \; u + v + w \leq 1.$$

We will need the Jacobian

$$\frac{d(u,v,w)}{d(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix} = -4.$$

It follows from this that

$$dxdydz = \frac{1}{4} dudvdw.$$

We can now compute the integral as follows

$$\begin{aligned}
&\frac{1}{4} \iiint_{\tilde{T}} uvw \, dudvdw \\
&\quad = \frac{1}{4} \int_0^1 u \, du \int_0^{1-u} v \, dv \int_0^{1-u-v} w \, dw \\
&\quad = \frac{1}{4} \int_0^1 u \, du \int_0^{1-u} \frac{v(1-u-v)^2}{2} \, dv \\
&\quad = \frac{1}{8} \int_0^1 u \, du \left(\left[-\frac{v(1-u-v)^3}{3} \right]_{v=0}^{v=1-u} + \int_0^{1-u} \int_0^{1-u} \frac{(1-u-v)^3}{3} \, dv \right) \\
&\quad = \frac{1}{8} \int_0^1 u \, du \left[-\frac{(1-u-v)^4}{12} \right]_{v=0}^{v=1-u} \\
&\quad = \frac{1}{8 \cdot 12} \int_0^1 u(1-u)^4 \, du \\
&\quad = \frac{1}{8 \cdot 12} \left(\left[-\frac{u(1-u)^5}{5} \right]_0^1 + \int_0^1 \frac{(1-u)^5}{5} \, du \right) \\
&\quad = \frac{1}{8 \cdot 12 \cdot 5 \cdot 6} = \frac{1}{2880}.
\end{aligned}$$