1. a) The Hessian $H$ must be positive semidefinite. Necessarily,

$$\det H = -(a + 2)^2 \geq 0 \iff a = -2.$$ 

For this value of $a$ we can complete the squares

$$x^T H x = (x_1 - 2x_2 + x_3)^2 \geq 0, \forall x \in \mathbb{R}^3,$$

hence, the matrix is positive semidefinite by definition.

Answer: $a = -2$.

b) If the minimum exists then it is a stationary point of $f$. The function is convex, therefore, a stationary point is the global minimum. Thus, the minimum exists $\iff$ a stationary point exists $\iff$ there exists a solution to

$$\nabla f(x, y, z) = 2(x + y + z) \begin{bmatrix} 1 & a \\ 1 & b \\ 1 & c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff a = b = c.$$

Answer: $a = b = c$.

c) See the book, Lemma 1, p. 43.

2. a) The function is convex iff $-\ln(y - x^2)$ is convex. Try $h(x, y) = x^2 - y$ convex, $g(t) = -\ln |t|$, $t < 0$, increasing ($g'(t) = -\frac{1}{t} > 0$) and convex ($g''(t) = \frac{1}{t^2} > 0$), hence, $g(h(x, y))$ is convex.

Answer: Yes.

b) Yes, $q_e \to +\infty$ when $(x, y)$ approaches the boundary $y = x^2$.

c) The minimum of $q_e$ on $D_q$ exists because the function is convex and there is a feasible stationary point

$$\nabla q_e = \begin{bmatrix} 1 + \frac{2x}{y-x^2} \\ 1 - \frac{1}{y-x^2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{cases} y - x^2 + 2x\epsilon = 0, \\ y - x^2 - \epsilon = 0. \end{cases} \iff \begin{cases} x = -\frac{1}{2}, \\ y = \frac{1}{4} + \epsilon, \end{cases}$$

which is the minimum of $q_e$. It converges to $(-1/2, 1/4)$, hence, it is the global minimum of the constrained problem.

3. a) The dual problem is

$$\min (6y_1 + 7y_2 + 2y_3) \quad \text{subject to} \quad \begin{cases} y_1 + 3y_2 + 3y_3 \geq 1, \\ 3y_1 + 5y_2 + y_3 \geq 4, \\ y_1 + y_2 + y_3 \geq 3, \\ y_2 \geq 0, y_3 \leq 0. \end{cases}$$

The CSP gives $y_2 = y_3 = 0$ and $y_1 + y_2 + y_3 = 3$, that is $y = (3, 0, 0)$. It is dual feasible, hence, both solutions are optimal.
b) The statement 2 is equivalent to

\[ A^T y = 0, \ b^T y > 0 \]

has no solution. Rewriting it as the standard (first) Farkas alternative

\[
\begin{bmatrix} A^T \\ -A^T \end{bmatrix} y \leq 0, \quad b^T y > 0
\]

makes the second alternative

\[
\begin{bmatrix} A \\ -A \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = b, \quad \begin{bmatrix} u \\ v \end{bmatrix} \geq 0
\]

being solvable. Denote \( x = u - v \) to conclude that it is equivalent to the statement 1.

4. a) Since all \( g_k \) are convex we have

\[
g(\lambda z_1 + (1 - \lambda) z_2) = \begin{bmatrix} g_1(\lambda z_1 + (1 - \lambda) z_2) \\ g_2(\lambda z_1 + (1 - \lambda) z_2) \\ \vdots \\ g_n(\lambda z_1 + (1 - \lambda) z_2) \end{bmatrix} \leq \begin{bmatrix} \lambda g_1(z_1) + (1 - \lambda) g_1(z_2) \\ \lambda g_2(z_1) + (1 - \lambda) g_2(z_2) \\ \vdots \\ \lambda g_n(z_1) + (1 - \lambda) g_n(z_2) \end{bmatrix} = \lambda g(z_1) + (1 - \lambda) g(z_2).
\]

Now we have convexity of \( h \) by definition from

\[
\begin{align*}
\frac{f(g(\lambda z_1 + (1 - \lambda) z_2))}{h(\lambda z_1 + (1 - \lambda) z_2)} & \leq f(\lambda g(z_1) + (1 - \lambda) g(z_2)) \leq \lambda f(g(z_1)) + (1 - \lambda) f(g(z_2)). \tag{1}
\end{align*}
\]

where (1) follows from the above relation and \( f \) being coordinate-wise increasing, and (2) follows from convexity of \( f \).

b) The function \( h_1 \) is convex since

\[
h_1(x, y, z) = \sqrt{x^6 + y^6 + z^6} = \sqrt{(|x|^3)^2 + (|y|^3)^2 + (|z|^3)^2}
\]

can be split as \( h = f \circ g \) where

\[
f(g_1, g_2, g_3) = \sqrt{g_1^2 + g_2^2 + g_3^2} = \|g\|, \quad g(x, y, z) = (|x|^3, |y|^3, |z|^3).
\]

Here all \( g_k(x, y, z) \) are convex and \( f \) is convex and coordinate-wise increasing.

The function \( h_2 \) is not convex. Set \( y = z = 0 \) to get the restriction \( \sqrt[3]{|x|} \) that is not convex.

c) The Hessian is

\[
H = \begin{bmatrix} 2 & 4y \\ 4y & 4x + 12(1 + a)y^2 \end{bmatrix}.
\]

It is positive-semidefinite in the set iff

\[
\det H = 8(x + (1 + 3a)y^2) \geq 0, \quad \forall x \geq 0.
\]

It is equivalent to \( 1 + 3a \geq 0 \iff a \geq -\frac{1}{3} \).

Answer: \( a \geq -\frac{1}{3} \).
5. Existence of the minimum.
Use \( x + y = 4 \) and \( x, y > 0 \) to conclude that \( x, y \) are bounded from above.
Take \( x = y = 2, \ z = 1/4 \). Then it is enough to consider \( xy + 2 \ln z \leq 4 - 2 \ln 4 \leq 2 \implies \ln z \leq 1 \), hence \( z \) is bounded from above as well.
Now use \( xyz \geq 1 \) to prove that \( x, y, z \) are bounded away from 0 from below. It makes the set compact. The minimum exists by Weierstrass theorem.

We set \( X = \{(x, y, z) : x > 0, \ y > 0, \ z > 0\} \),
\[
g(x, y, z) = 1 - xyz, \quad h(x, y, z) = x + y - 4.
\]

CQ points. No such.

KKT points.
\[
\begin{align*}
    y - uyz + v &= 0, \quad (1) \\
x - uxz + v &= 0, \quad (2) \\
\frac{2}{z} - uxy &= 0, \quad (3) \\
u(1 - xyz) &= 0, \quad (4) \\
u &\geq 0, \quad (5) \\
\text{feasibility} \quad (6)
\end{align*}
\]

\( u = 0 \) makes (3) impossible. For \( u > 0 \) we get \( xyz = 1 \) and \( u = 2 \) from (3). The first two equations yields
\[
v = y(2z - 1) = x(2z - 1) \implies (x - y)(2z - 1) = 0.
\]

If \( x = y \) then \( x + y = 4 \implies x = y = 2 \) and \( xyz = 1 \implies z = 1/4 \), hence, \((2, 2, 1/4)\) is a KKT point with \( f = 4 - 2 \ln 4 = 4 - 4 \ln 2 \).
If \( 2z = 1 \iff z = 1/2 \) then \( xy = 2 \) from (3) which together with \( x + y = 4 \) implies that \( x, y \) solves the quadratic equation
\[
t^2 - 4t + 2 = 0 \iff t = 2 \pm \sqrt{2}.
\]

Another KKT point is \((2 \pm \sqrt{2}, 2 \mp \sqrt{2}, 1/2)\), \( f = 2 - 2 \ln 2 \).
Answer: \((2 \pm \sqrt{2}, 2 \mp \sqrt{2}, 1/2)\).

6. a) Write down the Lagrange function
\[
L(x, u) = x^2 - 12x + y^2 + 2y + u_1(x^2 + y - 4) + u_2(-x^2 + y^2 + 1) = \]
\[
= (1 + u_1 - u_2) x^2 - 12x + (1 + u_2) y^2 + (2 + u_1) y + u_2 - 4u_1.
\]

Minimization w.r.t. \( y \geq 0 \) is clearly at \( y = 0 \).
Minimization w.r.t. \( x \) depends on:
if \( 1 + u_1 - u_2 \leq 0 \) then \( \inf_x L = -\infty \).
if \( 1 + u_1 - u_2 > 0 \) then the function \( L \) is convex in \( x \) and
\[
L'_x = 2(1 + u_1 - u_2)x - 12 = 0 \iff x = \frac{6}{1 + u_1 - u_2},
\]

hence it is the minimum. It gives

$$\Theta(u) = \begin{cases} \frac{36}{1 + u_1 - u_2} - 4u_1 + u_2, & \text{if } 0 \leq u_2 < 1 + u_1, \\ -\infty & \text{otherwise.} \end{cases}$$

Let us see if there is a stationary point w.r.t. $u_1$

$$\Theta'_{u_1} = \frac{36}{(1 + u_1 - u_2)^2} - 4 = 0 \iff 1 + u_1 - u_2 = 3 \iff u_1 = u_2 + 2.$$  

This $u_1$ maximizes $\Theta$ as the dual function is concave. It is left to maximize w.r.t. $u_2 \geq 0$ for the found $u_1$

$$\Theta = -12 - 4(u_2 + 2) + u_2 = -20 - 3u_2.$$  

The function is decreasing, hence, the maximum is at $u_2 = 0$. Therefore, $\bar{u}_2 = 0$, $\bar{u}_1 = 2$ and $\Theta(\bar{u}) = -20$. The candidate $x, y$ from above is $\bar{x} = 6/3 = 2$, $\bar{y} = 0$ gives $f(\bar{x}, \bar{y}) = -20 = \Theta(\bar{u})$, thus, the optimal point is $(2, 0)$.

b)

$$\Theta(u, v) = \inf_{x \in X} L(x, u, v) \leq L(x, u, v) = f(x) + u^T g(x) + v^T h(x) \leq f(x).$$