1. a) Exercise 4.8, p. 128.
   
   b) Lemma 1, p. 43.
   
   c) The implication is true. The proof is similar to Exercises 4.14–4.17. The system
   \[
   \begin{bmatrix}
   1 & 1 & 1 \\
   2 & -1 & 1 \\
   1 & 2 & -1
   \end{bmatrix}
   \lambda =
   \begin{bmatrix}
   3 \\
   5 \\
   1
   \end{bmatrix}
   \]
   has the solution \( \lambda = [2, 0, 1]^T \geq 0 \). Use Farkas.

2. a) The minimum is at the stationary point \((1, -1)\) as the function is convex.
   
   b) One possible auxiliary function is
   \[
   q_\mu(x, y) = f(x, y) + \mu \max\{0, -y\}^2.
   \]
   Since the optimal point in 2a) is infeasible and is taken as a starting point, all the iterations are going to be infeasible as well (see the discussion on the page 317), i.e. \( \max\{0, -y\}^2 = y^2 \) and, hence, \( q_\mu(x, y) = f(x, y) + \mu y^2 \). The stationary point
   \[
   \left( \frac{2\mu + 1}{4\mu - 1}, -\frac{3}{4\mu - 1} \right) \rightarrow \left( \frac{1}{2}, 0 \right) \leftarrow \mu \rightarrow +\infty.
   \]
   Convexity of \( q_\mu \) and Theorem 1, p. 316 approve optimality of the limit.

3. a) The dual problem is
   \[
   \max(c_1y_1 + c_2y_2) \quad \text{subject to} \quad \begin{cases}
   y_1 + 3y_2 \leq 3, \\
y_1 + y_2 \leq 1, \\
2y_1 + y_2 \leq 3, \\
y_2 \geq 0.
   \end{cases}
   \]
   Draw the dual feasible set (blue) and a level set of the dual function with
   \( c = (1, 2) \) (red). The optimal point is \( y = (0, 1) \).

The CSP for this dual solution gives \( x_3 = 0, x_1 + x_2 = 1 \) and \( 3x_1 + x_2 = 2 \),
with the primal solution being \( x = (1/2, 1/2, 0) \).
b) The set of all vectors $c$ that the dual problem has maximum at $(0, 1)$ is the dual cone (see p. 144 for definition) to the cone of all feasible directions at $A$.

![Diagram showing all possible c vectors](image)

Remark: the problem was ill-formulated (it says *primal* instead of *dual*). All reasonable solution attempts are granted the full credits.

4. a) Convex as a composition of linear $h(x, y) = x + y$ and convex $f(t) = \max\{t^2, t^3\}$.

b) Convex. $2x \cdot 4y^2 = 2x^2 + 2y^2$. Composition of convex and increasing $2^t$ and convex $x + 2y^2$.

c) The quadratic form $x^T H x = (x_1 + x_2 - x_3)^2 + (a - 1)x_3^2$ is convex if and only if $a \geq 1$ (i.e. if and only if $H$ is positive semidefinite). Hence, for $a \geq 1$ the level subset $\{x^T H x \leq 1\}$ is convex too. On the other hand, if $a < 1$ then the set is not convex since the intersection of the set with the line $x_1 = 2$, $x_2 = x_3 = t$ is not convex. It is easy to see that the intersection satisfies $4 + (a - 1)t^2 \leq 1$, which is true for large $|t|$ (as $a - 1$ is negative) and not true for $t = 0$. A counterexample is then the points $(2, t, t)$ and $(2, -t, -t)$ that are in the set, but the midpoint is not. Combining the two arguments gives the answer: $a \geq 1$.

5. Prove that the minimum exists. The origin is a CQ point. Two KKT points $(\pm \sqrt{2}, 1, 1)$ that are the minimum points.

6. a) The dual function is

$$
\Theta(u) = \begin{cases} 
- \frac{4}{u + 1} - \frac{(3 - u)^2}{4}, & \text{if } 0 \leq u \leq 1, \\
-\infty, & \text{otherwise}.
\end{cases}
$$

The derivative $\Theta'(u)$ is positive on $[0, 1]$, hence, the maximum of $\Theta$ is at $\bar{u} = 1$. For this $\bar{u}$ we have $\bar{y} = \bar{z} = 1$. To find $\bar{x}$ we have to use the CSP

$$
\bar{u}(\bar{z} - \bar{x}^2 + \bar{y}^2) = 0
$$

which gives $\bar{x} = \pm \sqrt{2}$. Check no duality gap to confirm the solution.

b) Exercise A.2b).