1. a) False, it converges in \( n \) steps (the dimension of \( x \)).
   b) False, e.g. \( g(x) = x^2, f(t) = -t, \) and \( f(g(x)) = -x^2 \) is not convex.
   c) True.
   d) True.
   e) False, e.g. \( f(x) = x \) lacks the global minimum.

2. a) For \( f(x, y) \): SD is good (circular level curves), NM is bad (\( H \) is singular).
   For \( g(x, y) \): SD is bad (ill-conditioned problem), NM is good (quadratic function).
   b) For \( d_k = -\nabla f(x_k) \): \( \min_\lambda f(x_k + \lambda d_k) \Rightarrow \frac{d}{d\lambda} f(x_k + \lambda d_k) = 0 \) at \( \lambda = \lambda_{\text{min}} \Rightarrow 0 = \nabla f(x_k + \lambda_{\text{min}} d_k)^T d_k = \nabla f(x_{k+1})^T d_k = -d_{k+1}^T d_k \Rightarrow d_{k+1} \perp d_k. \)

3. a) The dual problem is
   \[
   \max (2y_2 + y_3) \quad \text{subject to} \quad \begin{cases}
   5y_1 + 2y_2 + y_3 & \leq 3, \\
   3y_1 + y_2 - y_3 & \leq -5, \\
   6y_1 + y_2 - 2y_3 & \leq -10, \\
   -2y_1 + y_3 & \leq 4, \\
   y_1 & \geq 0, \ y_2 & \leq 0.
   \end{cases}
   \]
   The given \( x = (0, 0, 2, 5) \) is primal feasible. CSP gives then \( y_1 = 0, \) and the third and the fourth inequalities in the dual set becomes equalities, i.e.
   \[
   \begin{cases}
   y_2 - 2y_3 = -10, \\
   y_3 = 4.
   \end{cases} \Leftrightarrow y_2 = -2.
   \]
   Thus, \( y = (0, -2, 4) \) (feasible). No gap in the primal and the dual objective values (both are zeros) ensures optimality.
   b) Use Farkas. \( a \in [3, 4]. \)

4. a) 
   - \( f \) is not convex (set \( x = y = 0 \) and differentiate to see \( e^{\sqrt{z}} \) is not convex).
   - \( g \) is convex (max of two convex functions).
   - \( h = \|(1, x, y, z)\| \) is convex (by the definition of convex function + triangle inequality).
b) Complete the squares

\[ f(x, y, z) = x^2 + ay^2 + z^2 + 2xy + 2yz = (x + y)^2 + (y + z)^2 + (a - 2)y^2. \]

If \( a \geq 2 \) then \( f \) is convex and the set is convex as the sublevel set of a convex function. If \( a < 2 \) then the set is not convex (the intersection of the set with the line \( x = 2 - y, z = -y \) is not convex).

Answer: \( a \geq 2 \).

5. The problem is convex (\( x^3 \) is convex on \( x \geq 0 \)). The case when \( u > 0 \) has a KKT point \( (0, -1/4, -1/4) \), hence, it is the global minimizer.

6. a) The dual function is

\[ \Theta(v) = \begin{cases} v - 2v^2, & \text{if } v \geq 0, \\ v - 2v^2 + 2v\sqrt{-v}, & \text{if } v < 0. \end{cases} \]

The maximum of \( \Theta(v) \) is attained when \( v \geq 0 \) and the optimal \( \bar{v} = \frac{1}{4} \). Together with \( \bar{x} = 0, \bar{y} = \bar{z} = -\frac{1}{4} \) we get no duality gap, hence, the global minimum.

b) See the lecture notes, Lecture 4, page 2.