Lund University

Written Examination
Discrete Mathematics
Wednesday, January 11, 2012

Centre for Mathematical Sciences
Mathematics, Faculty of Science

No aids are allowed. Use only the department's paper and write only on one side of each sheet. Fill in the cover form completely and write your initials on each sheet. Give clear and short motivations to your solutions.

1. The characteristic polynomial associated with the homogeneous equation $a_{n+2}+4a_n = 0$ is $P(r) = r^2 + 4$, with complex roots $r_1 = 2i$ and $r_2 = -2i$. Therefore, the general solution of the homogeneous equation is

$$a_n^{(h)} = 2^n(\alpha \cos(\frac{\pi n}{2}) + \beta \sin(\frac{\pi n}{2})).$$

In order to find a particular solution of the non-homogeneous equation, we try with a function of the formula

$$a_n^p = (An + B)2^n$$

After some computations, we obtain $A = 1$ and $B = -1$ which offers the solution

$$a_n = 2^n(\alpha \cos(\frac{\pi n}{2}) + \beta \sin(\frac{\pi n}{2})) + (n-1)2^n.$$

Finally, applying the initial conditions we have $\alpha = 2$ and $\beta = 1$ and so

$$a_n = 2^n(2 \cos(\frac{\pi n}{2}) + \sin(\frac{\pi n}{2}) + n - 1).$$

2. We need to calculate the coefficient of x^{15} in the following generating function

$$(x^2 + x^3 + \cdots + x^{10})^4.$$

We notice that such coefficient coincides with the coefficient of x^{15} in the function

$$f(x) = (x^2 + x^3 + \cdots)^4 = x^8(1 + x + \cdots)^4 = x^8 \frac{1}{(1-x)^4}$$

Then, we need to find out the coefficient of x^7 in the function

$$\frac{1}{(1-x)^4} = \sum_{k=0}^{\infty} (-1)^k \binom{-4}{k} x^k.$$

This coefficient of x^7 is then

$$-\binom{-4}{7} = \binom{10}{7} = 120.$$

Please, turn over!

3. From Chinese remainder theorem, we have that the solution is given by

$$x = \sum_i a_i x_i N_i.$$

with $N_1 = 42$, $N_2 = 35$, $N_3 = 30$, $a_1 = 2$, $a_2 = 5$, $a_3 = 1$. Now, we need to calculate the factors x_i which satisfy $x_i N_i \equiv 1 \pmod{n_i}$. By the euclidean algorithm, we have $x_1 = 3$, $x_2 = 5$ and $x_3 = 4$ wich finally offers the solution

$$x = 2 \cdot 3 \cdot 42 + 5 \cdot 5 \cdot 35 + 1 \cdot 4 \cdot 30 = 1247 \equiv 197 \pmod{210}.$$

4. **a)** The solution is given by $\frac{11!}{3!2!2!2!} = 831600$.

b) The solution is given by $\frac{4!}{2!} \cdot \frac{8!}{3!2!2!} = 20160$.

c) We use the exclusion/inclusion principle to solve the problem. We denote by c_1 the property of PARK appears and by c_2 the property of ORK appears. Then the required quantity coincides with

$$N(\bar{c_1}\bar{c_2}) = N - N(c_1) - N(c_2) + N(c_1 c_2).$$

Now

$$\begin{aligned} N &= 831600, \\ N(c_1) &= \frac{8!}{2!} = 20160, \\ N(c_2) &= \frac{9!}{3!2!} = 30240, \\ N(c_1 c_2) &= \frac{6!}{2!} = 360. \end{aligned}$$

Then,

$$N(\bar{c_1}\bar{c_2}) = 781560.$$

5. We prove that the second equation has no solution and therefore, the system cannot have a solution. Let's assume that the second equation has a solution x. Then, by definition, there exist $y \in \mathbb{Z}$ such that $3x = 7+9y$. This implies $7 = 3x-9y = 3(x-3y)$ which is a contradiction since 7 and 3 are relatively prime numbers.

The example does not contradict Chinese remainder theorem since it does not satisfy its hypotheses.

6. The code words defined by the generator matrix are

$$C = E(W) = \{000000, 001111, 010011, 011100, 100110, 101001, 110101, 111010\}.$$

We notice that C is a group code and that the minimum weight of the non-zero elements of C is 3. Then, we deduce that the separation of this linear code is 3 and then, we can detect up to double errors or correct up to single errors.

None of the received word are a code words. We calculate the syndrome of the first received word and obtain 101, which does not coincide with any column of H. Therefore, it can't be properly decoded, since it was the consequence of multiple errors.

On the other hand, the syndrome of the second received word is 111 which coincides with the third column of H. Therefore, a single error in transmission happened in the third coordinate and so, so, it's associated with the code word $c = 101110 + 001000 = 100110$. Then, finally, the received word can be decoded as $w = 100$.