No books, notes, computational devices, etc. are allowed. Use clear handwriting and give clear careful motivations. All answers should be fully simplified, but may contain factorials or powers. In particular they should not contain binomial coefficients or Stirling numbers. Fill in the form completely and write your personal identifier on each sheet of paper.

1. The characteristic equation has 2 as a double root which implies that the homogeneous equation has the solutions $A \cdot 2^n + B \cdot n \cdot 2^n$. Adding the particular solution $3n$ and applying the initial conditions we find that $A = 6$ and $B = -3$. Hence $a_n = 6 \cdot 2^n - 3 \cdot n \cdot 2^n + 3n$.

2. 
   a) Each element has a unique representation $[a + bx]$ with $a, b \in \mathbb{Z}_{11}$, so there are 121 elements.
   
   b) If $R$ is a field or not depends on whether $h(x) = x^2 + 1$ is irreducible. If it is reducible it must have a zero by the factor theorem, but it is easy to check that none of the element in $\mathbb{Z}_{11}$ are a zero. Hence $R$ is a field.
   
   c) We get $[x + 7] \cdot [x + 2] = [x^2 + 9x + 14] = [x^2 + 9x + 3 - (x^2 + 1)] = [9x + 2]

3. 
   a) For each object we choose a container, resulting in $3^7$ possibilities.
   
   b) This equals the number of strings you can make using two separators and seven circles, that is $\left( \binom{9}{2} \right) = 36$.
   
   c) This is given by the Stirling number $S(7, 3) = 301$. (To compute the value use the recurrence $S(n+1, k) = S(n, k-1) + kS(n, k)$ to create the relevant part of Stirling’s triangle.)
   
   d) Dividing into cases depending on the number of non-empty containers we get $S(7, 3) + S(7, 2) + S(7, 1) = 301 + 63 + 1 = 365$

4. 
   a) The Chinese remainder theorem guarantees that the system has a unique solution modulo 154, as 2, 7 and 11 are pairwise relatively prime. Solving the subsystems with two LHS’s replaced by zero at a time we get $x_1 = 77, x_2 = 44$ and $x_3 = 42$ resulting in the solution $x = x_1 + x_2 + x_3 = 163 = 154 + 9$. All solutions are given by $9 + 154k$ where $k$ is any integer.
   
   b) Here the fact that 2 and 14 have a common factors implies that the Chinese remainder theorem does not apply to the system of three equations. It does however apply to the first pair of equations and we find the solutions of this subsystem to be $9 + 14m$. All these solutions satisfy the third equations and therefore the answers is $9 + 14m$ where $m$ is any integer.

Turn the page!
5. a) For an \([n,m]\)-code the control matrix is \((n - m) \times n\). Thus \(m\), the dimension of the code is \(6 - 4 = 2\) and the number of words \(7^2 = 49\).

b) Words of the form \(abccba\) are automatically perpendicular to rows 1 and 3 of \(H\). Being perpendicular to the other two rows correspond to satisfying a system of two linear equations. Solving them we find that \(abccba = kkkkkk\) for any \(k \in \mathbb{Z}_7\). We may take for example \(w_1 = 000000\), \(w_2 = 111111\) and \(w_3 = 222222\). Then all pairwise Hamming distances are 6.

c) From the structure of \(H\) we know that \(C\) is a Reed-Solomon code with separation five. (All 4x4 subdeterminants of \(H\) are Vandermonde determinants and therefore known to be non-zero. This means any four columns of \(H\) are independent. Any five columns are dependent as they are in a four-dimensional space. Thus the separation must be five by theorem 3.10 in Andersson.)

6. These problems can be solved using exponential generating functions.

a) We use the exponential generating function

\[
g(x) = (e^x - 1)^2 \frac{(e^x - e^{-x})}{2} (e^x)^2
\]

where the first factor counts red and blue blocks, the second green blocks and the third white and yellow blocks.

\[
g(x) = \frac{1}{2}(e^{5x} - 2e^{4x} + 2e^{2x} - e^x).
\]

We search 10! times the coefficient of \(x^{10}\), and by Taylor expansion we obtain

\[
\frac{5^{10} - 2 \cdot 4^{10} + 2 \cdot 2^{10} - 1}{2}
\]

b) In this case the exponential generating function is

\[
f(x) = (e^x - 1)^5 = e^{5x} - 5e^{4x} + 10e^{3x} - 10e^{2x} + 5e^x - 1.
\]

Our answer, 10! times the coefficient of \(x^{10}\), is \(5^{10} - 5 \cdot 4^{10} + 10 \cdot 3^{10} - 10 \cdot 2^{10} + 5\).

(Alternatively note that this is also the number of surjective functions from a set of 10 elements onto a set of 5 elements given by \(5!S(10, 5)\).)