1. a) Selecting four out of six students can be done in \( \binom{6}{4} = 15 \) ways.

b) The queue can be formed in \( 6 \cdot 5 \cdot 4 \cdot 3 = 360 \) ways.

c) The number of distributions equals the number of strings you can form using 10 circles (=apples) and 5 separators, i.e., \( \binom{15}{5} = 3003 \)

d) Four choices for each student results in \( 4^6 = 4096 \) possibilities.

2. Using Fermat’s little theorem we find that in \( \mathbb{Z}_5 \) we have:

\[ 3^{44} = 1, \quad 2^{16} = 1, \quad 4^{80} = 1 \]

and \( 2^{41} = 2 \). Then in each case eliminate \( y \) in the second equation by subtracting two times the first equation. Now we have:

a) \[
\begin{align*}
2x + y &= 2 \\
-3x &= -1
\end{align*}
\]

\[ x = (-1)(-3)^{-1} = (-1)(-2) = 2 \]

\[ y = 2 - 2x = 2 - 2 \cdot 2 = -2 \]

b) \[
\begin{align*}
2x + y &= 2 \\
0 &= -1
\end{align*}
\]

Thus the first system of equations has one solution \((x, y) = (2, 3)\) while the second system has no solutions.

3. First we note that if a number is not divisible by 2 then it follows that it is not divisible by 4 either. Thus we can count numbers not divisible by 2, 3 or 5. We do this using the principle of inclusion/exclusion. Let \( c_i \) be the condition divisible by \( i \) for \( i = 2, 3, 5 \). The ground set is \( S = \{2, 3, 4, \cdots 999, 1000\} \) and the number of elements not satisfying any condition equals

\[
S_0 - S_1 + S_2 - S_3 = |S| - (N(c_2) + N(c_3) + N(c_5)) + (N(c_2c_3) + N(c_2c_5) + N(c_3c_5))
\]

\[
-N(c_2c_3c_5) = 999 - \left( \left\lfloor \frac{1000}{2} \right\rfloor + \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{5} \right\rfloor \right) + \left( \left\lfloor \frac{1000}{6} \right\rfloor + \left\lfloor \frac{1000}{10} \right\rfloor + \left\lfloor \frac{1000}{15} \right\rfloor \right)
\]

\[
-\left\lfloor \frac{1000}{30} \right\rfloor = 999 - 1033 + 332 - 33 = 265
\]

(Since the number one is not counted in any of \( S_1, S_2 \) or \( S_3 \) we can compute them in the same ways as if \( S \) included the number 1.)

4. Consider the linear code \( C \) over \( \mathbb{Z}_7 \) with generating matrix

\[
G = \begin{pmatrix}
1 & 2 & 2 & 0 & -5 \\
0 & 1 & 3 & -3 & -1
\end{pmatrix}
\]

a) Since \( C \) is 2-dimensional over a field of size 7 it contains \( 7^2 = 49 \) words.
b) By transforming $G$ to standard form we find that

$$H = \begin{pmatrix}
4 & -3 & 1 & 0 & 0 \\
-6 & 3 & 0 & 1 & 0 \\
3 & 1 & 0 & 0 & 1
\end{pmatrix}$$

is a control matrix.

c) The dual code $C^\perp$ is generated by the rows of the control matrix. One possible answer is $(4 - 3 1 0 0)$, $(-6 3 0 1 0)$, $(3 1 0 0 1)$ and $(-2 0 1 1 0)$.

d) Computing the syndromes $H^tw_1 = (0 0 0)$, $H^tw_2 = (0 0 0)$ and $H^tw_3 = (2 1 3)$ we find that $w_1$ and $w_2$ are code words while $w_3$ is not.

5. Assume that you want to fill a wall space of size $2 \times k$ (where $k$ is a positive integer) with tiles of size $1 \times 2$.

a) Let $a_k = c_k + d_k$ where $c_k$ are the number of tilings ending with one vertical tile where $d_k$ counts tilings ending with a pair of horizontal tiles. Then $c_k = a_{k-1}$ and $d_k = a_{k-2}$ since before the end tile(s) we can have any pattern of length $k - 1$ or $k - 2$ in the respective cases. As initial conditions we can use $a_0 = 1$ and $a_1 = 1$.

b) By the same reasoning as above we get $b_k = c_k + d_k = 2b_{k-1} + 4b_{k-2}$. The reason we get $2b_{k-1}$ here is that given a pattern of size $k - 1$ we can end it in two ways, using a black tile or using a white tile. In the case $d_k$ there are four colourings of the ending tiles. As initial conditions we can use $b_0 = 1$ and $b_1 = 2$.

c) Solving the recurrence using standard methods we find that

$$b_k = \frac{1}{2\sqrt{5}}((1 + \sqrt{5})^{k+1} - (1 - \sqrt{5})^{k+1})$$

6. Let $a(x) = x^2 + 1$ and $b(x) = x^2 - x$. The polynomials $a$ and $b$ are relatively prime and using the Euclidean algorithm we find that $1 = (\frac{x}{2} - \frac{1}{2})b(x) + (1 - \frac{x}{2})a(x)$. It follows that $x(\frac{x}{2} - \frac{1}{2})b(x) + 2(1 - \frac{x}{2})a(x) = \frac{x^4}{2} - 2x^3 + \frac{5x^2}{2} - x + 2$ works as $p$. (By adding multiples of $a(x)b(x)$ you get other possible $p(x)$. Lowest possible degree $p$ is $-\frac{3}{2}x^3 + 2x^2 - \frac{x}{2} + 2$.)