

You may use any books and computer programs (e.g. Matlab and Maple), but it is not permitted to get help from other persons. For Problem 8, Matlab is required.

Hand in solutions to 6 of 9 problems below. For a passing grade (3), at least 3 problems have to be solved correctly. Credits can be given for partially solved problems. Write your solutions neatly and explain your calculations.

The exam can be picked up between 9/3 and 13/3. The exam should be handed in to the Student office in the Mathematics Department, LTH at the latest exactly 7 days after collecting the exam. Write your name, section-year (or subject for PhD-students), id-number and email address on the first page, and write your name on each of the following pages. I will contact you about the oral part of the exam when the written exam papers have been marked.

Problems can have some parameters ( $a, b, c, d \dots$ ). The value of the parameters should be chosen according to your birthday in the format  $yymmdd=abcdef$ . For example if you are born on March 12, 1991 then  $a = 9, b = 1, c = 0, d = 3, e = 1, f = 2$  in your problems.

1. Solve the following linear programming problem

$$\max \mathbf{c}^T \mathbf{x}$$

with  $\mathbf{c}^T = [-1 \ a \ b \ d]$  subject to  $\mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq 0$ , where

$$\mathbf{A} = \begin{bmatrix} 4 & -2 & c & 5 \\ 1 & 4 & -5 & 6 \\ 2 & -25 & 4 & 5 \\ 7 & 1 & -4 & 0 \\ 2 & 0 & -6 & 1 \\ 0 & 3 & -3 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 12 \\ 25 \\ -10 \\ 28 \\ -4 \\ 10 \end{bmatrix},$$

using the two phase method. Present the initial and final tableaux for each phase. Are there any feasible solutions? If there is an optimal solution, find it.

2. Using the definition of the dual of a problem in standard form, find the dual of the linear programming problem

$$\max(\mathbf{c}^T \mathbf{x} + \mathbf{d}^T \mathbf{x}')$$

subject to

$$\mathbf{Ax} + \mathbf{Bx}' \leq \mathbf{b},$$

$$\mathbf{x} \geq \mathbf{0},$$

$$\mathbf{x}' \in \mathbb{R}.$$

3. Maximize  $z = x_1 + 2x_2 + x_3 + x_4$ , subject to

$$\begin{aligned} 2x_1 + x_2 + 3x_3 + x_4 &\leq 8, \\ 2x_1 + 3x_2 + 4x_4 &\leq 12, \\ 3x_1 + x_2 + 2x_3 &\leq 18, \\ x_j &\geq 0, \\ x_j &\in \mathbb{Z}, \quad j = 1, 2, 3, 4, \end{aligned}$$

using the cutting plane method. Present the initial and final tableau for each LP problem solved, and explain how the cutting planes are derived.

4. Assume that  $\mathbf{a}_1, \dots, \mathbf{a}_m$  are given nonzero vectors in  $\mathbb{R}^3$  and that  $b_1, \dots, b_m$  are given positive numbers. Let

$$\mathbb{P} = \{\mathbf{x} \in \mathbb{R}^3; \mathbf{a}_i^T \mathbf{x} \leq b_i, i = 1, \dots, m\}.$$

You can think of  $\mathbb{P}$  as a region in  $\mathbb{R}^3$  whose “walls” are formed by the planes

$$P_i = \{\mathbf{x} \in \mathbb{R}^3; \mathbf{a}_i^T \mathbf{x} = b_i\}, \quad i = 1, \dots, m.$$

Suppose that we want to find the center and the radius of the largest sphere contained in  $\mathbb{P}$ . Formulate this as a linear programming problem. Use the fact that the distance  $d(y, P_i)$  of a point  $y \in \mathbb{R}^3$  to the plane  $P_i$  is given by the formula

$$d(y, P_i) = \frac{|b_i - \mathbf{a}_i^T y|}{\|\mathbf{a}_i\|}.$$

5. An automobile manufacturer has to assembly plants in three different locations with capacities of 100K, 120K and 60K cars per year, respectively (where K=1000). The cars are assembled and then sent to major markets, in total, four different locations, each location requiring 50K, 40K, 90K and 70K of cars per year, respectively. The price (in EUROS) per car transported to each location is as follows.

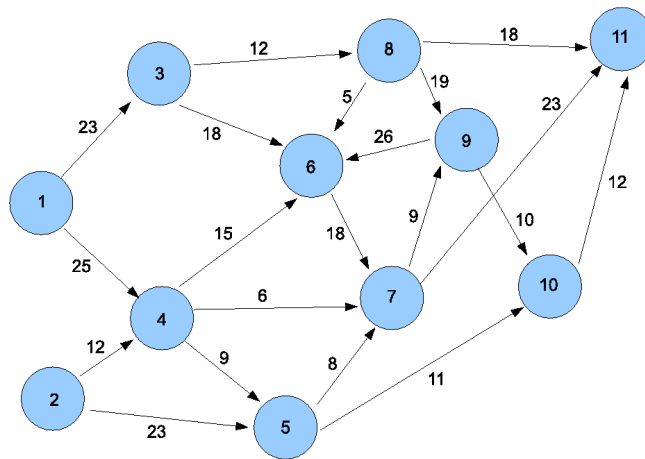
		market			
		1	2	3	4
plant	1	11	*	8	6
	2	7	5	6	8
	3	7	6	8	5

However, there are no routes from plant 1 to market 2 (and therefore no price is given in the table).

a) Find the optimal transportation pattern in order to minimize costs.

b) Due to limited transportation capacity, no more than 40K cars can be transported from plant 1 to market 3. Similarly, no more than 40K cars from plant 2 to market 3 and no more than 40K cars from plant 3 to market 3 can be transported. Find the optimal transportation pattern in order to minimize costs with these restrictions.

6. The organisation TrikiLeaks has stored mirrored information on two servers in Sydney and Jakarta. These are represented by node 1 and 2 in the graph below. They want to transfer this information as quickly as possible to Paris (node 11 in the graph). The capacities of the different parts of the network are given in the graph below. What is the maximum capacity that information can be sent from Sydney and Jakarta to Paris? At the same time, a foreign power wants to stop TrikiLeaks from sending the information. How should they sabotage the network with *minimum* effort so that all connection from Sydney and Jakarta to Paris is cut off? Assume that the effort of sabotaging one connection is proportional to the capacity.



7. A consulting firm has 6 consultants. The following table shows the maximum number of days each of these consultants can work:

$C1$	$C2$	$C3$	$C4$	$C5$	$C6$
250	200	$150 + 10d$	220	300	$100 + 10b$

The firm has 10 clients, each of them needs 120 days of consulting. The following table shows how much they agree to pay for one day of work of the corresponding consultant.

	$C1$	$C2$	$C3$	$C4$	$C5$	$C6$
1	210	200	350	250	100	300
2	200	220	320	230	160	340
3	180	210	300	240	150	330
4	190	220	250	210	130	310
5	210	200	300	190	100	300
6	180	210	330	250	180	320
7	200	220	320	220	170	310
8	190	200	210	200	180	300
9	180	240	320	240	160	290
10	190	250	310	270	140	310

How should the company organize the consulting to optimize the profit? One consultant can work for different clients but not the same day. Several consultants can work for the same client.

8. Suppose that a data sequence

$$y_0, \dots, y_N$$

is given. In MATLAB, an estimate of the *spectral density*  $\Phi$  of the signal  $y = (y_0, \dots, y_N)$  is given by

$$\text{» Phi} = \text{abs(fft(y)).}^2;$$

(Roughly speaking, the spectral density gives the energy content of the signal  $y$  at the frequency  $\theta$ .) This estimate is referred to as the *periodogram estimate* of the sound signal  $y$ . It is common to use decibels (dB) to express the spectral density of sound signals. So one defines

$$\Psi = 10 \log_{10} \Phi.$$

Now the task is to fit a rational model to the estimated spectral density  $\Psi$  above, that is we want to find a  $\hat{\Psi}$ , such that  $\hat{\Psi}$  is close to  $\Psi$ , and moreover,

$$\hat{\Psi}(\theta) = \frac{S(\theta)}{T(\theta)}, \quad \theta \in [0, 2\pi],$$

where

$$\begin{aligned} S(\theta) &= p_0 + p_1 \cos \theta + \dots + p_n \cos(n\theta), \\ T(\theta) &= 1 + q_1 \cos \theta + \dots + q_n \cos(n\theta). \end{aligned}$$

The aim is that the long data  $y$  (which has  $N$  components), is now replaced by the short data of coefficients  $p_0, \dots, p_n$  and  $q_1, \dots, q_n$ , and if  $2n + 1 \ll N$ , we have saved a lot of memory space. This idea is used in speech processing for characterizing short speech segments. The point is that if we compare two samples of a person uttering the same word twice, then the data in the time domain is quite different. But the frequency domain picture is very similar, and an approximation of the frequency domain picture is enough for reproduction of that sound. Thus it makes sense to approximate the  $\Psi$ . In this manner, the rational model can be used for speech recognition, speaker recognition, speech compression and so on.

So what one would like to do ideally is minimize some norm of the difference between  $\Psi$  and  $\hat{\Psi}$ , that is  $\|\Psi - \hat{\Psi}\|$ . But right now we want to use linear programming to solve the problem. So instead of this, we consider the minimization of norm of

$$T(\Psi - \hat{\Psi}) = T\Psi - S.$$

We will specify the norm below.

To begin with, for our  $y$ , we will use a standard sound signal in Matlab. This sound signal is called *handel*, which is a sample from a choir singing Händel's Messiah. In order to get the periodogram estimate  $\Psi$  for this  $y$ , we proceed as follows:

```
» load handel;
» N = 1024;
» Psi = 10*log10((1/N)*abs(fft(y(10001:10000+N))).^2);
» Psi = Psi(1:N/2);
```

(One can listen to the  $y$  we are using by using the command `sound(y,Fs)`. Since our signal  $y$  is a real sequence, it is enough to take the first  $\frac{N}{2}$  components of  $\Psi$ ; this is the reason for the  $N/2$  in the fourth MATLAB command above.)

The above sequence of commands in MATLAB gives us a vector with values  $\Psi(\theta_k)$ , where

$$\theta_k = \frac{k\pi}{\frac{N}{2}}, \quad k = 0, \dots, \frac{N}{2} - 1.$$

We will use the  $\ell^1$ -norm for sequences. Thus if we have a sequence  $f = (f_k)_{0 \leq k \leq \frac{N}{2}-1}$ , then

$$\|f\|_1 := \sum_{k=0}^{\frac{N}{2}-1} |f_k|.$$

Since we are considering the problem of minimization of the norm of  $\Psi T - S$ , we are led to the following optimization problem:

$$(P) : \text{minimize } \left\| (\Psi(\theta_k) \cdot T(\theta_k) - S(\theta_k))_{0 \leq k \leq \frac{N}{2}-1} \right\|_1. \quad (1)$$

The variables in the above minimization problem are the coefficients  $p_0, \dots, p_n$  and  $q_1, \dots, q_n$ .

**a)** Recast the optimization problem (P) in (1) as a linear programming problem.

Motivate your formulation, that is, explain why your linear programming problem enables one to solve the original optimization problem (P). In your linear programming problem, you should specify clearly the variables, the constraints and the objective function.

**b)** Use the Optimization toolbox in MATLAB to solve your linear programming problem. Use  $n = 8$ . The MATLAB command `help linprog` displays a help text. The problem

$$\begin{cases} \text{minimize} & c^T x, \\ \text{subject to} & Ax \leq b, \end{cases}$$

can be solved using the simplex method using the following commands:

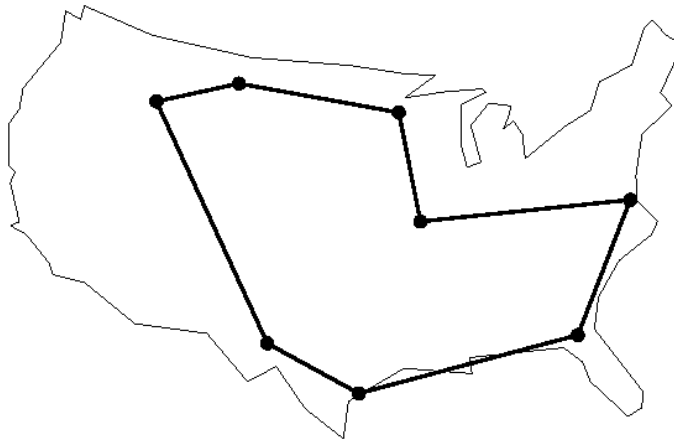
```
» options=optimset('Simplex','on','Largescale','off','Display','iter')
» [xfval,exitflag,output]=linprog(c,A,b,[],[],[],[],[],options);
```

**c)** Plot  $(\Psi(\theta_k) \cdot T(\theta_k))_{0 \leq k \leq \frac{N}{2}-1}$  and  $(S(\theta_k))_{0 \leq k \leq \frac{N}{2}-1}$  in the same graph. Write any observations you have.

**d)** Plot  $(\Psi(\theta_k))_{0 \leq k \leq \frac{N}{2}-1}$  and  $(\widehat{\Psi}(\theta_k))_{0 \leq k \leq \frac{N}{2}-1}$  in the same graph. Write any observations you have.

**e)** Repeat the problem for some other segment of the sound signal  $y$ . For example, instead of loading `handel`, one could use `chirp`, `laughter`, or `gong`.

9. Senator J. Caiman is planning his schedule for the presidential race in the autumn. He would like to visit as many cities as possible, but at the same time, he is getting old and would prefer to travel as little as possible. If he were to visit only 8 major cities (see map below), he has sketched a possible route with a total distance of 2849 miles. Is this optimal? Certainly the senator needs to visit more cities to attract the American voters all over the country. Help the senator to schedule his route in order to minimize total distance for 25 cities (see map on next page). At position  $(i, j)$  in the matrix  $C$ , the distance in miles is given from city  $i$  to city  $j$ . He should start and end his route in the same city. The distance data can be downloaded from the course homepage: `tsp8.mat`, `tsp15.mat` and `tsp25.mat`.



Solve the above instance of the traveling salesman problem with the **branch and bound** method using the following specifications.

**Bound:** A lower bound of the objective function can be obtained by solving the following assignment problem:

$$\min \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} \quad (2)$$

$$\sum_{i=1}^n x_{ij} = 1 \text{ for all } j \quad (3)$$

$$\sum_{j=1}^n x_{ij} = 1 \text{ for all } i \quad (4)$$

where  $x_{ij}$  is 1 if the route includes the connection between city  $i$  and city  $j$ , otherwise  $x_{ij}$  is 0. (Here it is assumed that  $c_{ii}$  is set to a large number for  $i = 1, \dots, n$  so that  $x_{ii} = 0$  for all solutions.) If the solution corresponds to a tour, we are done. However, the assignment may result in infeasible solutions of the type  $x_{i_1 i_2} = x_{i_2 i_3} = \dots = x_{i_m i_1} = 1$ , where  $m < n$ .

**Branch:** Suppose a subproblem results in an infeasible solution and consider the smallest cycle such that  $x_{i_1 i_2} = \dots = x_{i_m i_1} = 1$ . This problem can be branched into  $m$  subproblems: for each branch, add one (and only one) of the constraints  $x_{i_1 i_2} = 0, \dots, x_{i_m i_1} = 0$ , since not all of these connections can be part of a tour. Hint: Instead of adding the constraint  $x_{ij} = 0$  to a subproblem, one can equivalently set  $c_{ij}$  to a large number.

- a) Motivate why the assignment problem gives a lower bound on the problem. Why are solutions of the assignment problem of the type:  $x_{i_1 i_2} = x_{i_2 i_3} = \dots = x_{i_m i_1} = 1$ , where  $m < n$ , infeasible for the traveling salesman problem?
- b) It may take too long time to solve the full problem of 25 cities. Try to solve it for 8 and 15 cities first, and then 25 cities.
- c) If you did not solve it for all 25 cities, give upper and lower bounds on the optimal distance for the full problem.

