The Rogers-Ramanujan Identities and their Generalizations

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Abstract

The Rogers-Ramanujan identities were first discovered and proved by L.J. Rogers, and appear as corollaries to more general results in a paper published by him in 1894. They were later rediscovered and proved by both S. Ramanujan and I. Schur.

In this paper we present three proofs to the Rogers-Ramanujan identities. The first two proofs are due to Rogers and Ramanujan and are both highly inspired by Rogers’ work from 1894. The third one is due to Schur and it is a combinatorial proof.

A generalization of the Rogers-Ramanujan identities for all moduli is also stated and proved. For this we use generalizations of the identities due to B. Gordon, G.E. Andrews and D.M. Bressoud.

We end the paper with a short discussion regarding the theta function’s involvement in the proofs of the mentioned identities.
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1 Introduction

The Rogers-Ramanujan identities are

\[ \prod_{n=0}^{\infty} \frac{1}{(1 - q^{5n+1})(1 - q^{5n+3})} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1 - q)(1 - q^2) \ldots (1 - q^n)} \]

and

\[ \prod_{n=0}^{\infty} \frac{1}{(1 - q^{5n+2})(1 - q^{5n+3})} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1 - q)(1 - q^2) \ldots (1 - q^n)}, \]

where \(|q| < 1\). Hardy has expressed that "it would be difficult to find more beautiful formulae than the 'Rogers-Ramanujan' identities," and briefly described their early history in [8] with the following passage:

"...They were rediscovered nearly 20 years later by Mr Ramanujan, who communicated them to me in a letter from India in February 1913. Mr Ramanujan had then no proof of the formulae, which he had found by a process of induction. I communicated them in turn to Major MacMahon and to Prof. O. Perron of Tübingen; but none of us were able to suggest a proof; and they appear, unproved, in Ch. 3, Vol. 2, 1916, of Major MacMahon’s Combinatory Analysis."

Ramanujan, who rediscovered the identities, later came across Rogers’ article from 1894 (see [9]). This discovery resulted in a correspondence between Rogers and Ramanujan and later in a joint publication in 1919 (see [8]). About the same time Schur discovered the identities independently and presented two proofs in 1917 (see [10]), both different from Rogers’ and Ramanujan’s.

In Section 3 of this paper we present three of the above discussed proofs and further comment on them. Section 4 is dedicated to generalizations of the Rogers-Ramanujan identities. By using Gordons generalization (see [7]), Andrews’ generalization (see [5]) and Bressoud’s generalization (see [6]) we show that the Rogers-Ramanujan identities have analogs for all moduli.

When studying both the proofs of the Rogers-Ramanujan identities and the proofs of the generalizations of the identities a reappearance of the theta function is noticeable. We therefore end this paper with Section 5, where we define theta functions and elliptic functions and prove some fundamental properties of elliptic functions in the form of four propositions.
Finally, I would like to express my utmost gratitude and thanks to my supervisor prof. Arne Meurman for his guidance, patience and editorial support.
2 Preparatory Work

Before addressing the main topic of this essay useful theorems will be introduced and proved. We begin with a theorem due to Cauchy, continue with a corollary due to Euler and finish with a theorem of great importance due to Jacobi.

In the sequel the following abbreviations will be used

\[(x; q)_n = \begin{cases} \prod_{k=1}^{n}(1 - xq^{k-1}) & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases}\]

and

\[(x; q)_\infty = \lim_{n \to \infty} (x; q)_n = \prod_{k=1}^{\infty}(1 - xq^{k-1}).\]

**Theorem 2.1 (Cauchy)** Let \(|q| < 1\) and \(|t| < 1\). Then

\[\prod_{n=0}^{\infty} \frac{(1 - xtq^n)}{(1 - tq^n)} = 1 + \sum_{n=1}^{\infty} \frac{(x; q)_n t^n}{(q; q)_n}.\]

**Proof.** Define

\[f(t) = \prod_{n=0}^{\infty} \frac{(1 - xtq^n)}{(1 - tq^n)}.\]

Since the infinite product is uniformly convergent for \(|t| < 1\) (\(x\) and \(q\) fixed) \(f\) is analytic on the open unit disc. We therefore know that there exist \(c_n = c_n(x, q)\) such that

\[f(t) = \sum_{n=0}^{\infty} c_n t^n,\]

where \(c_0 = f(0) = 1\). Now

\[(1 - t)f(t) = (1 - t) \prod_{n=0}^{\infty} \frac{(1 - xtq^n)}{(1 - tq^n)} = (1 - xt) \prod_{n=1}^{\infty} \frac{(1 - xtq^n)}{(1 - tq^n)} = (1 - xt)f(qt),\]

which gives us

\[\sum_{n=0}^{\infty} c_n t^n - \sum_{n=0}^{\infty} c_n t^{n+1} = \sum_{n=0}^{\infty} c_n q^n t^n - \sum_{n=0}^{\infty} c_n xq^n t^{n+1}.\]

Comparing the coefficients of \(t^n\) for \(n > 0\) in the above relation we get

\[c_n - c_{n-1} = q^n c_n - xq^{n-1} c_{n-1}.\]
From this we then have
\[
c_n = \frac{1 - xq^{n-1}}{1 - q^n} c_{n-1} = \frac{1 - xq^{n-1}}{1 - q^n} \cdot \frac{1 - xq^{n-2}}{1 - q^{n-1}} c_{n-2} = \ldots = \frac{(1 - xq^{n-1})(1 - xq^{n-2}) \ldots (1 - x)}{(1 - q^n)(1 - q^{n-1}) \ldots (1 - q)} c_0 = \frac{(x; q)_n}{(q; q)_n},
\]
and hence,
\[
f(t) = \sum_{n=0}^{\infty} \frac{(x; q)_n t^n}{(q; q)_n},
\]
the result follows.

**Remark** Theorem 2.1 is known as the $q$-analog of the binomial series. The reason becomes apparent when we set $x = q^m$, $m \in \mathbb{N}$. We then have
\[
1 + \sum_{n=1}^{\infty} \frac{(1 - q^m)(1 - q^{m+1}) \ldots (1 - q^{m+n-1})}{(1 - q)(1 - q^2) \ldots (1 - q^n)} t^n = \prod_{n=0}^{\infty} \frac{(1 - tq^{n+m})}{(1 - tq^n)},
\]
which tends to
\[
1 + \sum_{n=1}^{\infty} \binom{m+n-1}{n} t^n = \frac{1}{(1 - t)^{m}}
\]
as $q$ tends to $1^-$. ▲

**Corollary 2.1 (Euler)** Let $|q| < 1$ and $|t| < 1$. Then
\[
(i) \quad \prod_{n=0}^{\infty} (1 - tq^n)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{m^n}{(q; q)_n},
\]
\[
(ii) \quad \prod_{n=0}^{\infty} (1 + tq^n) = 1 + \sum_{n=1}^{\infty} \frac{\ln^2(q; q)_n}{(q; q)_n}.
\]

**Proof.** The result in (i) follows directly by setting $x = 0$ in Theorem 2.1. For (ii) replace $x$ by $x/a$ and $t$ by $at$ in Theorem 2.1. By Theorem 2.1 we then have the following equality
\[
\prod_{n=0}^{\infty} \frac{1 - xtq^n}{1 - atq^n} = 1 + \sum_{n=1}^{\infty} \frac{(a - x)(a - xq) \ldots (a - xq^{n-1})}{(q; q)_n} t^n
\]
for $|at| < 1$. Now by setting $x = -1$ and $a = 0$ in the above equation we obtain the second part of the stated theorem. □

The below stated is an almost immediate consequence of Euler’s result and is therefore perhaps better viewed as a corollary. However, since it is of great importance in what shall come we label it a theorem.
Theorem 2.2 (The Jacobi Triple Product) Let $|q| < 1$ and $x \neq 0$.

Then

$$\sum_{n=-\infty}^{\infty} x^n q^{n^2} = \prod_{n=0}^{\infty} (1 - q^{2n+2})(1 + xq^{2n+1})(1 + x^{-1}q^{2n+1}).$$

Proof. Let $|q| < |x|$. By the second part of Euler’s corollary we then have

$$\prod_{n=0}^{\infty} (1 + xq^{2n+1}) = \prod_{n=0}^{\infty} (1 + (xq)(q^2)^n) = 1 + \sum_{n=1}^{\infty} \frac{(xq)^{n}(q^2)^{\frac{1}{2}n(n-1)}}{(q^2; q^2)_n}$$

$$= \frac{1}{(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} x^n q^{n^2}(q^{2n+2}; q^2)_\infty. \tag{1}$$

The last equality holds since $(q^{2n+2}; q^2)_\infty = 0$ for $n$ negative. Again by the second part of Corollary 2.1 we have

$$(q^{2n+2}; q^2)_\infty = \prod_{m=0}^{\infty} (1 - q^{2n+2}q^{2m}) = \prod_{m=0}^{\infty} (1 + (-q^{2n+2})(q^2)^{m})$$

$$= 1 + \sum_{m=1}^{\infty} \frac{(-q^{2n+2})^m(q^2)^{\frac{1}{2}m(m-1)}}{(q^2; q^2)_m}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2+2mn+m}}{(q^2; q^2)_m}. \tag{2}$$

Now, by combining (1) and (2) we obtain

$$\prod_{n=0}^{\infty} (1 + xq^{2n+1}) = \frac{1}{(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} x^n q^{n^2} \left( \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2+2mn+m}}{(q^2; q^2)_m} \right)$$

$$= \frac{1}{(q^2; q^2)_\infty} \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^m x^{n+m} q^{(n+m)^2} x^{-m} q^m}{(q^2; q^2)_m}$$

$$= \frac{1}{(q^2; q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m x^{-m} q^m}{(q^2; q^2)_m} \sum_{n=-\infty}^{\infty} x^{n+m} q^{(n+m)^2}$$

$$= \frac{1}{(q^2; q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-x^{-1}q)^m}{(q^2; q^2)_m} \sum_{n=-\infty}^{\infty} x^n q^{n^2}.$$

By the first part of Euler’s corollary we know

$$\sum_{m=0}^{\infty} \frac{(-x^{-1}q)^m}{(q^2; q^2)_m} = \prod_{m=0}^{\infty} \frac{1}{(1 + x^{-1}q^{2m})}.$$
and therefore we have
\[ \prod_{n=0}^{\infty} (1 + xq^{2n+1}) = \frac{1}{(q^2; q^2)_\infty} \cdot \frac{1}{(-x^{-1}q; q^2)_\infty} \sum_{n=-\infty}^{\infty} x^n q^{n^2}, \]
which gives us
\[ \sum_{n=-\infty}^{\infty} x^n q^{n^2} = \prod_{n=0}^{\infty} (1 - q^{2n+2})(1 + xq^{2n+1})(1 + x^{-1}q^{2n+1}). \]
Since the same argument holds for \(|q| < |x^{-1}|\) the theorem follows. \qed
3 The Rogers-Ramanujan Identities

This section will focus on presenting proofs to the Rogers-Ramanujan identities. The identities were stated in the introduction of this essay but are also stated in the theorem below using the abbreviations defined in Section 2.

Theorem 3 (The Rogers-Ramanujan Identities)

Let \(|q| < 1\). Then

\[
\begin{align*}
(i) & \quad \prod_{n=0}^{\infty} \frac{1}{(1-q^{m+1})(1-q^{m+n})} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{n}}, \\
(ii) & \quad \prod_{n=0}^{\infty} \frac{1}{(1-q^{m+2})(1-q^{m+n})} = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{n}}.
\end{align*}
\]

Remark The combinatorial interpretation of identity (i) states that the number of partitions of an integer \(N = \sum_{j=1}^{m} \alpha_j\) with \(\alpha_j \geq \alpha_{j+1} + 2, j \geq 1\), is equal to the number of partitions of \(N = \sum_{j=1}^{m} \tilde{\alpha}_j\) with \(\tilde{\alpha}_j \equiv \pm 1 \pmod{5}\), \(j \geq 1\).

Similarly, identity (ii) can be interpreted combinatorially as saying that the partitions of an integer \(N = \sum_{j=1}^{m} \alpha_j\) with \(\alpha_j > 1\) and \(\alpha_j \geq \alpha_{j+1} + 2, j \geq 1\), are equinumerous with the partitions of \(N = \sum_{j=1}^{m} \tilde{\alpha}_j\) with \(\tilde{\alpha}_j \equiv \pm 2 \pmod{5}\), \(j \geq 1\). \(\blacksquare\)

In the three subsections that follow we will present three proofs of the Rogers-Ramanujan identities, i.e. three proofs of Theorem 3. The first proof is by Rogers, and it is the one from the article he jointly published with Ramanujan in 1919 (see [8], pp. 212-214). In the second subsection we find Ramanujan’s proof from the same article (see [8], pp. 214-216). As we will see the two proofs are in principle the same, but the details differ. In the last and third subsection we present Schur’s combinatorial proof (see [10], pp. 307-312).

3.1 Proof by Rogers

Define

\[
V_m(x, q) = \sum_{n=0}^{\infty} (-1)^n x^{2n} q^{2n^2+\frac{1}{2}n(n+1)-mn} (1-x^m q^{2mn}) c_n, \tag{3}
\]

where \(c_n = \frac{(x,q)_n}{(q,q)_n}\) and \(c_n = 0\) for \(n < 0\). Furthermore define the operator \(\eta\) by \(\eta f(x) = f(xq)\). The below presented lemma holds for all \(m, n\) but will later only be used in the two special cases \(m = 1, 2\).
Lemma 3.1.1 Let \(|q| < 1\). Then
\[
\frac{V_n(x, q) - V_{m-1}(x, q)}{1 - x} = x^{m-1}qV_{3-m}(x, q).
\]

Proof. To begin with we have
\[
(1 - x)\eta c_{n-1} = (1 - x)\eta \frac{(x; q)_{n-1}}{(q; q)_{n-1}} = (1 - x)\frac{(xq; q)_{n-1}}{(q; q)_{n-1}}
= (1 - q^n)\frac{(x; q)_n}{(q; q)_n} = (1 - q^n)c_n, \tag{4}
\]
and
\[
(1 - x)\eta c_n = (1 - x)\eta \frac{(x; q)_n}{(q; q)_n} = (1 - x)\frac{(xq; q)_n}{(q; q)_n}
= (1 - xq^n)\frac{(x; q)_n}{(q; q)_n} = (1 - xq^n)c_n. \tag{5}
\]
Furthermore we have
\[
\frac{V_m(x, q) - V_{m-1}(x, q)}{1 - x} = \frac{1}{1 - x} \sum_{n=0}^{\infty} (-1)^n x^{2n} q^{2n^2 + \frac{1}{2}n(n+1) - mn} (1 - xq^{2m})c_n
= \frac{1}{1 - x} \sum_{n=0}^{\infty} (-1)^n x^{2n} q^{2n^2 + \frac{1}{2}n(n+1) - (m-1)n} (1 - xq^{2m-n})c_n
= \frac{1}{1 - x} \sum_{n=0}^{\infty} (-1)^n x^{2n} q^{2n^2 + \frac{1}{2}n(n+1) - mn} (1 - q^n + x^{m-1}q^{2m-1}n(1 - xq^n))c_n.
\]
Now by replacing \((1 - q^n)c_n\) and \((1 - xq^n)c_n\) according to (4) and (5) respectively and by remembering that \(c_n = 0\) for \(n < 0\) we have
\[
\frac{V_m(x, q) - V_{m-1}(x, q)}{1 - x} = \sum_{n=0}^{\infty} (-1)^n x^{2n} q^{2n^2 + \frac{1}{2}n(n+1) - mn}(\eta c_{n-1} + x^{m-1}q^{(2m-1)n}\eta c_n)
= \sum_{n=0}^{\infty} (-1)^n x^{2n} q^{2n^2 + \frac{1}{2}n(n+1) - mn}\eta c_{n-1}
+ \sum_{n=0}^{\infty} (-1)^n x^{2n} q^{2n^2 + \frac{1}{2}n(n+1) - mn} x^{m-1}q^{(2m-1)n}\eta c_n =
\]
\[
\sum_{n=0}^{\infty} (-1)^{n+1} x^{2(n+1)} q^{2(n+1)^2 + \frac{1}{2} n(n+1)(n+2) - m(n+1)} \eta c_n
\]

\[
+ \sum_{n=0}^{\infty} (-1)^n x^{2n} q^{2n^2 + \frac{1}{2} n(n+1) - mn} x^{m-1} q^{(2m-1)n} \eta c_n
\]

\[
= \sum_{n=0}^{\infty} (-1)^{n+1} x^{2n} q^{2n^2 + \frac{1}{2} n(n+1) - mn} x^{m-1} q^{(2n+3) - m} \eta c_n
\]

\[
+ \sum_{n=0}^{\infty} (-1)^n x^{2n} q^{2n^2 + \frac{1}{2} n(n+1) - mn} x^{m-1} q^{(2n+3) - m} \eta c_n
\]

\[
= \sum_{n=0}^{\infty} (-1)^n x^{2n} q^{2n^2 + \frac{1}{2} n(n+1) - mn} (x^{m-1} q^{(2m-1)n} - x^2 q^{5n+3-m}) \eta c_n
\]

\[
= x^{m-1} \sum_{n=0}^{\infty} (-1)^n (xq)^{2n} q^{2n^2 + \frac{1}{2} n(n+1) - (3-m)n} (1 - x^3 q^{6n+3-2mn-m}) \eta c_n
\]

\[
= x^{m-1} \sum_{n=0}^{\infty} (-1)^n (xq)^{2n} q^{2n^2 + \frac{1}{2} n(n+1) - (3-m)n} (1 - x^3 q^{6n+3-2mn-m}) \eta c_n
\]

\[
= x^{m-1} \eta \sum_{n=0}^{\infty} (-1)^n x^{2n} q^{2n^2 + \frac{1}{2} n(n+1) - (3-m)n} (1 - x^3 q^{6n+3-2mn-m}) c_n.
\]

Since

\[
V_{3-m}(x, q) = \sum_{n=0}^{\infty} (-1)^n x^{2n} q^{2n^2 + \frac{1}{2} n(n+1) - (3-m)n} (1 - x^3 q^{6n+3-2mn-m}) c_n
\]

the result follows. \[\blacksquare\]

Define

\[
v_m(x, q) = \frac{1}{(x, q)_\infty} V_m.
\]

**Lemma 3.1.2** Let \(|q| < 1\). Then

(i) \(v_1(x, q) = \eta v_2(x, q)\),

(ii) \(v_2(x, q) - v_1(x, q) = x \eta v_1(x, q)\).
Proof. By Lemma 3.1.1 we have the following
\[
v_m(x, q) - v_{m-1}(x, q) = \frac{1}{(x; q)_\infty} (V_m(x, q) - V_{m-1}(x, q))
\]
\[
= \frac{1 - x}{(x; q)_\infty} \frac{V_m(x, q) - V_{m-1}(x, q)}{1 - x}
\]
\[
= \frac{1}{(xq; q)_\infty} (x^{m-1} \eta V_{3-m}(x, q))
\]
\[
= x^{m-1} \eta \frac{1}{(x; q)_\infty} V_{3-m}(x, q)
\]
\[
= x^{m-1} \eta v_{3-m}(x, q). 
\]  
(7)

For identity (i) set \( m = 1 \) in (7). The result now follows since \( v_0(x, q) = 0, \)
\[
v_1(x, q) - v_0(x, q) = \eta v_2(x, q).
\]
For identity (ii) set \( m = 2 \) in (7).

Proof of Theorem 3. For the second Rogers-Ramanujan identity, i.e. identity (ii) in Theorem 3.1, set
\[
v_1(x, q) = \sum_{n=0}^{\infty} \alpha_n x^n = 1 + \alpha_1 x + \alpha_2 x^2 + \ldots,
\]
where \( \alpha_n = \alpha_n(q) \) and \( \alpha_0 = 1 \). Applying the operator \( \eta \) to the second identity in Lemma 3.1.2 we obtain
\[
\eta(v_2(x, q) - v_1(x, q)) = \eta x \eta v_1(x, q)
\]
which gives us
\[
\eta v_2(x, q) - \eta v_1(x, q) = x q \eta^2 v_1(x, q).
\]
Then by the first identity in Lemma 3.1.2 we have
\[
v_1(x, q) - \eta v_1(x, q) = x q \eta^2 v_1(x, q).
\]
So we have
\[
\sum_{n=0}^{\infty} \alpha_n x^n - \sum_{n=0}^{\infty} \alpha_n q^n x^n = \sum_{n=0}^{\infty} \alpha_n q^{2n+1} x^{n+1}.
\]
Now, by comparing the coefficients of \( x^n \) in the above relation we see
\[
(1 - q^n) \alpha_n = q^{2n-1} \alpha_{n-1},
\]
which is equivalent to
\[
\alpha_n = \frac{q^{2n-1}}{1 - q^n} \alpha_{n-1}.
\]
and therefore gives us

\[
\alpha_n = \frac{q^{2n-1}}{(1 - q^n)^{\alpha_{n-1}}} = \frac{q^{2n-1}}{(1 - q^n)} \cdot \frac{q^{2n-3}}{(1 - q^{n-1})^{\alpha_{n-2}}}
\]

\[
= \ldots = \frac{q^{2n-1}q^{2n-3} \ldots q}{(1 - q^n)(1 - q^{n-1}) \ldots (1 - q)^{\alpha_0}}
\]

\[
= q^{n^2} \frac{1}{(q; q)_n}.
\]

So,

\[
v_1(x, q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} x^n.
\]

(8)

An immediate consequence of the above result and (6) is

\[
\frac{1}{(x; q)_\infty} V_1(x, q) = \sum_{n=0}^{\infty} \frac{q^{n^2} x^n}{(q; q)_n}.
\]

By replacing \( x \) with \( q \) in the above we obtain the right hand side of identity (ii) of Theorem 3.1.

\[
\frac{1}{(q; q)_\infty} V_1(q, q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n}.
\]

(9)

By the definition of \( V_m(x, q) \) in (3) we have

\[
V_1(q, q) = \sum_{n=0}^{\infty} (-1)^n q^{2n^2 + \frac{1}{2} n(n+1)} (1 - q^{2n})
\]

\[
= \sum_{n=0}^{\infty} (-1)^n q^{2n^2 + \frac{1}{2} n(n+1)+n} (1 - q^{2n+1}).
\]

\( V_1(q, q) \) can be rewritten into a Laurent series in the following way

\[
V_1(q, q) = \sum_{n=0}^{\infty} (-1)^n q^{2n^2 + \frac{1}{2} n(n+1)+n} - \sum_{n=0}^{\infty} (-1)^n q^{2n^2 + \frac{1}{2} n(n+1)+n} q^{2n+1}
\]

\[
= \sum_{n=0}^{\infty} (-1)^n q^{2n^2 + \frac{1}{2} n(n+1)+n} - \sum_{n=1}^{\infty} (-1)^{n-1} q^{2(n-1)^2 + \frac{1}{2} n(n-1)+(n-1)} q^{2n-1}
\]

\[
= \sum_{n=0}^{\infty} (-1)^n q^{2n^2 + \frac{1}{2} n(n+1)+n} + \sum_{n=1}^{\infty} (-1)^n q^{2n^2 + \frac{1}{2} n(n-1)-n}
\]

\[
= \sum_{n=0}^{\infty} (-1)^n q^{2n^2 + \frac{1}{2} n(n+1)+n} + \sum_{n=-\infty}^{-1} (-1)^n q^{2n^2 + \frac{1}{2} n(n-1)+n}
\]

\[
= \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2 + \frac{1}{2} n(n+1)+n}. 
\]

(10)
Then by the Jacobi triple product we have

\[
V_1(q, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2 + \frac{1}{2}n^2 + \frac{1}{2}n + n} = \sum_{n=-\infty}^{\infty} (-q^{\frac{3}{2}})^n (q^{\frac{3}{2}})^n^2 \\
= \prod_{n=0}^{\infty} (1 - (q^{\frac{3}{2}})^{2n+2})(1 + (-q^{\frac{3}{2}})(q^{\frac{5}{2}})^{2n+1})(1 + (-q^{\frac{3}{2}})^{-1}(q^{\frac{5}{2}})^{2n+1}) \\
= \prod_{n=0}^{\infty} (1 - q^{5n+5})(1 - q^{5n+4})(1 - q^{5n+1}).
\] (11)

So by (9) and (11) the result follows

\[
\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \frac{1}{(q; q)_\infty} V_1(q, q) \\
= \frac{1}{(q; q)_\infty} \prod_{n=0}^{\infty} (1 - q^{5n+5})(1 - q^{5n+4})(1 - q^{5n+1}) \\
= \prod_{n=0}^{\infty} \frac{1}{(1 - q^{5n+2})(1 - q^{5n+3})}.
\]

For identity (i), by using the first part of Lemma 3.1.2, the result in (8) and noting that \(\eta^{-1} f(x, q) = f(xq^{-1}, q)\) is the inverse of \(\eta\), we obtain

\[
v_2(x, q) = \eta^{-1} v_1(x, q) = \eta^{-1} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} x^n = \sum_{n=0}^{\infty} \frac{q^{n^2-n}}{(q; q)_n} x^n.
\]

Using (6) and letting \(x = q\) in the above we obtain the left hand side of identity (i) in Theorem 3

\[
\frac{1}{(q; q)_\infty} V_2(q, q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n}. \tag{12}
\]

By rewriting \(V_2(q, q)\) into a Laguerre series as in (10) and then using the Jacobi triple product as before we have

\[
V_2(q, q) = \ldots = \prod_{n=0}^{\infty} (1 - q^{5n+5})(1 - q^{5n+3})(1 - q^{5n+2}). \tag{13}
\]

And then finally by combining the results in (12) and (13) we obtain

\[
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q)_\infty} V_2(q, q) \\
= \frac{1}{(q; q)_\infty} \prod_{n=0}^{\infty} (1 - q^{5n+5})(1 - q^{5n+3})(1 - q^{5n+2}) \\
= \prod_{n=0}^{\infty} \frac{1}{(1 - q^{5n+1})(1 - q^{5n+4})}.
\]
which is the first Rogers-Ramanujan identity.

3.2 Proof by Ramanujan

Define

\[ G(x, q) = 1 + \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{\frac{1}{2}n(5n-1)} (1 - xq^{2n}) \frac{(xq; q)_{n-1}}{(q; q)_n}, \]  

(14)

and

\[ H(x, q) = \frac{G(x, q)}{1 - xq} - G(xq, q). \]  

(15)

**Lemma 3.2.1** For \( G(x, q) \) and \( H(x, q) \) as in (14) and (15) we have

\[ H(x, q) = xq(1 - xq^2)G(xq^2, q). \]

Before proceeding to the proof of Lemma 3.2.1 we note that \( G(x, q) \) can be rewritten by associating the second part of each term with the first part of the succeeding term in the following way

\[
G(x, q) = 1 + \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{\frac{1}{2}n(5n-1)} (1 - q^n) + q^n (1 - xq^n) \frac{(xq; q)_{n-1}}{(q; q)_n} \\
= 1 + \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{\frac{1}{2}n(5n-1)} \frac{(xq; q)_{n-1}}{(q; q)_n} \\
+ \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{\frac{1}{2}n(5n-1)} q^n (1 - xq^n) \frac{(xq; q)_{n-1}}{(q; q)_n} \\
= 1 + \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{\frac{1}{2}n(5n-1)} \frac{(xq; q)_{n-1}}{(q; q)_n} \\
+ \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{\frac{1}{2}n(5n-1)} q^n \frac{(xq; q)_n}{(q; q)_n} \\
= 1 - x^2 q^2 + \sum_{n=2}^{\infty} (-1)^n x^{2n} q^{\frac{1}{2}n(5n-1)} \frac{(xq; q)_{n-1}}{(q; q)_n} \\
+ \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{\frac{1}{2}n(5n-1)} q^n \frac{(xq; q)_n}{(q; q)_n} \\
= (1 - x^2 q^2) - \sum_{n=1}^{\infty} (-1)^n x^{2(n+1)} q^{\frac{1}{2}(n+1)(5(n+1)-1)} \frac{(xq; q)_n}{(q; q)_n} \\
+ \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{\frac{1}{2}n(5n-1)} q^n \frac{(xq; q)_n}{(q; q)_n} =
\]
\[= (1 - x^2 q^2) - \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{2n(5n-1)} x^2 q^{5n+2} (xq; q)_n (q; q)_n \]
\[+ \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{2n(5n-1)} q^n (xq; q)_n (q; q)_n \]
\[= (1 - x^2 q^2) + \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{2n(5n-1)} q^n (1 - x^2 q^{2(2n+1)}) (xq; q)_n (q; q)_n \]
\[= \sum_{n=0}^{\infty} (-1)^n x^{2n} q^{2n(5n+1)} (1 - x^2 q^{2(2n+1)}) (xq; q)_n (q; q)_n. \quad (16)\]

Proof. By using (16) for the first term in (15) and (14) for the second term in (15) we obtain

\[H(x, q) = \]
\[= \frac{1}{1 - xq} \sum_{n=0}^{\infty} (-1)^n x^{2n} q^{2n(5n+1)} (1 - x^2 q^{2(2n+1)}) (xq; q)_n (q; q)_n \]
\[\quad - \left(1 + \sum_{n=1}^{\infty} (-1)^n (xq)^{2n} q^{2n(5n-1)} (1 - xq^{2n+1}) (xq^2; q)_{n-1} (q; q)_n \right) \]
\[= \frac{1}{1 - xq} + \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{2n(5n+1)} (1 - x^2 q^{2(2n+1)}) (xq^2; q)_{n-1} (q; q)_n \]
\[\quad - \left(1 + \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{2n(5n+1)} q^n (1 - xq^{2n+1}) (xq^2; q)_{n-1} (q; q)_n \right) \]
\[= xq + \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{2n(5n+1)} (1 - x^2 q^{2(2n+1)} - q^n + xq^{2n+1}) (xq^2; q)_{n-1} (q; q)_n \]
\[= xq + \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{2n(5n+1)} ((1 - q^n) + xq^{3n+1} (1 - xq^{n+1})) (xq^2; q)_{n-1} (q; q)_n. \]

Now, as before, by associating the second part of each term with the first part of the succeeding term,

\[H(x, q) = \]
\[= xq + \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{2n(5n+1)} (1 - q^n) (xq^2; q)_{n-1} (q; q)_n \]
\[\quad + \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{2n(5n+1)} xq^{3n+1} (1 - xq^{n+1}) (xq^2; q)_{n-1} (q; q)_n = \]
\[= \sum_{n=1}^{\infty} (-1)^n x^{2n} q^{\frac{1}{2} n(5n+1)} \frac{(xq^2; q)_n}{(q; q)_n} \]
\[+ \sum_{n=0}^{\infty} (-1)^n x^{2n} q^{\frac{1}{2} n(5n+1)} xq^{3n+1} \frac{(xq^2; q)_n}{(q; q)_n} \]
\[= \sum_{n=0}^{\infty} (-1)^{n+1} x^{2(n+1)} q^{\frac{1}{2} (n+1)(5(n+1)+1)} \frac{(xq^2; q)_n}{(q; q)_n} \]
\[+ \sum_{n=0}^{\infty} (-1)^n x^{2n} q^{\frac{1}{2} n(5n-1)} x^2 q^{6n+3} \frac{(xq^2; q)_n}{(q; q)_n} \]
\[= \sum_{n=0}^{\infty} (-1)^n x^{2n} q^{\frac{1}{2} n(5n-1)} xq^{4n+1} \frac{(xq^2; q)_n}{(q; q)_n} \]
\[= xq \sum_{n=0}^{\infty} (-1)^n (xq^2)^{2n} q^{\frac{1}{2} n(5n-1)} (1 - (xq^2)^{2n}) \frac{(xq^2; q)_n}{(q; q)_n} \]
\[= xq \left( 1 - xq^2 + \sum_{n=1}^{\infty} (-1)^{n} (xq^2)^{2n} q^{\frac{1}{2} n(5n-1)} (1 - (xq^2)^{2n}) \frac{(xq^2; q)_n}{(q; q)_n} \right) \]
\[= xq \left( 1 - xq^2 \right) \left( 1 + \sum_{n=1}^{\infty} (-1)^{n} (xq^2)^{2n} q^{\frac{1}{2} n(5n-1)} (1 - (xq^2)^{2n}) \frac{(xq^3; q)_{n-1}}{(q; q)_n} \right) \]
\[= xq(1 - xq^2)G(xq^2, q), \]

we obtain the result. \[\blacksquare\]

Define
\[F(x, q) = \frac{G(x, q)}{(xq; q)_\infty}. \quad (17)\]

**Lemma 3.2.2** For \(F(x, q)\) as in (17) we have
\[F(x, q) = F(xq, q) + xqF(xq^2, q).\]

**Proof.** By the definition of \(H(x, q)\) in (15) and by Lemma 3.2.1 we have
\[
\frac{G(x, q)}{1 - xq} - G(x, q) = xq(1 - xq^2)G(xq^2, q).
\]
Now, by substituting $G(x, q)$, $G(xq, q)$ and $G(xq^2, q)$ according to (17) we get the following

$$\frac{(xq; q)_\infty}{1 - xq} F(x, q) - (xq^2; q)_\infty F(xq, q) = xq(1 - xq^2)(xq^3; q)_\infty F(xq^2, q).$$

Since $(xq; q)_\infty/(1 - xq) = (1 - xq^2)(xq^3; q)_\infty = (xq^2; q)_\infty$ we obtain the result by dividing both sides by $(xq^2; q)_\infty$. ■

Proof of Theorem 3. Set

$$F(x, q) = \sum_{n=0}^\infty \beta_n x^n = 1 + \beta_1 x + \beta_2 x^2 + \ldots,$$

where $\beta_n = \beta_n(q)$ and $\beta_0 = 1$. By Lemma 3.2.2 we have

$$\sum_{n=0}^\infty \beta_n x^n = \sum_{n=0}^\infty \beta_n q^n x^n + xq \sum_{n=0}^\infty \beta_n q^{2n} x^n.$$

Comparing the coefficients of $x^n$ in the above we see that

$$\beta_n = \beta_n q^n + \beta_{n-1} q^{2n-1},$$

which is equivalent to

$$\beta_n = q^{2n-1} \frac{1}{1-q^n} \beta_{n-1}.$$

From this we then have

$$\beta_n = \frac{q^{2n-1}}{1-q^n} \beta_{n-1} = \frac{q^{2n-1}}{1-q^n} \frac{q^{2n-3}}{1-q^{n-1}} \beta_{n-2} = \ldots = \frac{q^{2n-1}q^{2n-3} \ldots q^1}{(1-q^n)(1-q^{n-1}) \ldots (1-q)} \beta_0 = \frac{q^{2n}}{(q; q)_n}.$$

Thus our $F(x, q)$ can be written in the following way

$$F(x, q) = \sum_{n=0}^\infty \frac{q^2}{(q; q)_n} x^n. \quad (19)$$

For identity (ii) of Theorem 3.1 set $x = q$. By (17) and (19) we then have

$$\frac{G(q, q)}{(q^2; q)_\infty} = \sum_{n=0}^\infty \frac{q^{n^2+n}}{(q; q)_n}. \quad (20)$$
Where, using (14), we see

\[ G(q, q) = 1 + \sum_{n=1}^{\infty} (-1)^n q^{2n} q^{\frac{1}{2} n(5n-1)} (1 - q^{2n+1}) \frac{q^{\frac{n}{2}} q^{n-1}}{(q^2; q)_n} \]

\[ = \sum_{n=0}^{\infty} (-1)^n q^{2n} q^{\frac{1}{2} n(5n-1)} (1 - q^{2n+1}) \frac{1}{1 - q} \]

\[ = \frac{1}{1 - q} \sum_{n=0}^{\infty} (-1)^n q^{2n} q^{\frac{1}{2} n(5n-1)} (1 - q^{2n+1}). \]

This can now be rewritten into a Laurent series in the following way

\[ G(q, q) = \frac{1}{1 - q} \left( \sum_{n=0}^{\infty} (-1)^n q^{2n} q^{\frac{1}{2} n(5n-1)} - \sum_{n=0}^{\infty} (-1)^n q^{2n} q^{\frac{1}{2} n(5n-1)} q^{2n+1} \right) \]

\[ = \frac{1}{1 - q} \left( \sum_{n=0}^{\infty} (-1)^n q^{2n} q^{\frac{1}{2} n(5n-1)} - \sum_{n=1}^{\infty} (-1)^{n-1} q^{2(n-1)} q^{\frac{1}{2} (n-1)(5(n-1)-1)} q^{2(n-1)+1} \right) \]

\[ = \frac{1}{1 - q} \left( \sum_{n=0}^{\infty} (-1)^n q^{2n} q^{\frac{1}{2} n(5n-1)} + \sum_{n=1}^{\infty} (-1)^n q^{-2n} q^{\frac{1}{2} n(5n+1)} \right) \]

\[ = \frac{1}{1 - q} \left( \sum_{n=0}^{\infty} (-1)^n q^{2n} q^{\frac{1}{2} n(5n-1)} + \sum_{n=-\infty}^{-1} (-1)^n q^{2n} q^{\frac{1}{2} n(5n-1)} \right) \]

\[ = \frac{1}{1 - q} \sum_{n=-\infty}^{\infty} (-1)^n q^{2n} q^{\frac{1}{2} n(5n-1)}. \quad (21) \]

Then by the Jacobi triple product we have

\[ (1 - q) G(q, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{5}{2} n^2 + \frac{5}{2} n} = \sum_{n=-\infty}^{\infty} (-q^\frac{5}{2})^n (q^2)^n q^2 \]

\[ = \prod_{n=0}^{\infty} (1 - (q^\frac{5}{2})^{2n+2})(1 - q^\frac{3}{2}(q^\frac{5}{2})^{2n+1})(1 - q^{-\frac{3}{2}}(q^\frac{5}{2})^{2n+1}) \]

\[ = \prod_{n=0}^{\infty} (1 - q^{5n+5})(1 - q^{5n+4})(1 - q^{5n+1}). \quad (22) \]

Now by (20) and (22) identity (ii) follows

\[ \sum_{n=0}^{\infty} q^{n^2 + n} \frac{(q; q)_n}{(q^2; q)_{\infty}} = \frac{G(q, q)}{(q^2; q)_{\infty}} = \prod_{n=0}^{\infty} (1 - q^{5n+5})(1 - q^{5n+1})(1 - q^{5n+4}) \]

\[ = \prod_{n=0}^{\infty} (1 - q^{5n+2})(1 - q^{5n+3}). \]
For identity (i) we set $x = 1$ in (17) and (18) and obtain

$$
\frac{G(1, q)}{(q, q)_\infty} = \sum_{n=0}^{\infty} \frac{q^n}{(q, q)_n}.
$$

(23)

Where, using the same method as in (21) and the Jacobi triple product, $G(1, q)$ can be rewritten in the following way

$$
G(1, q) = \ldots = \prod_{n=0}^{\infty} (1 - q^{5n+1})(1 - q^{5n+2})(1 - q^{5n+3}).
$$

(24)

By (23) and (24) we then have

$$
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q, q)_n} = \frac{G(1, q)}{(q, q)_\infty} = \prod_{n=0}^{\infty} \frac{(1 - q^{5n+5})(1 - q^{5n+2})(1 - q^{5n+3})}{(q, q)_\infty}

= \prod_{n=0}^{\infty} \frac{1}{(1 - q^{5n+1})(1 - q^{5n+4})},
$$

which proves the first identity of Theorem 3. □

Remark We have in the above presented proof omitted a result Ramanujan included in the original proof from 1919. The omitted part, although of no importance to the proof of Theorem 3, is both an interesting and important result.

Ramanujan defined a forth function

$$
K(x, q) = \frac{G(x, q)}{1 - xq}.
$$

(25)

Using definition (15) and Lemma 3.2.1 we see that

$$
K(x, q) = \frac{G(x, q)}{1 - xq} = \frac{1}{G(xq, q)} = \left( G(x, q) + H(x, q) \right) = \frac{1}{G(xq, q)}

= \frac{G(xq, q) + xq(1 - xq^2)G(xq^2, q)}{G(xq, q)}

= 1 + xq \frac{(1 - xq^2)G(xq^2, q))}{G(xq, q)} = 1 + \frac{xq}{K(xq, q)}.
$$

By repeating the above step we see that $K(x, q)$ can be expressed as a continued fraction

$$
K(x, q) = 1 + \frac{xq}{K(xq, q)} = 1 + \frac{xq}{1 + \frac{xq}{K(xq^2, q)}} = \ldots = 1 + \frac{xq xq^2 xq^3 \ldots}{1 + 1 + 1 + \ldots}.
$$

Now, by setting $x = 1$ in the definition of $K(x, q)$ and using the above we see

$$
\frac{1}{K(1, q)} = \frac{1}{G(1, q)}.
$$
From (22) and (24) we obtain the omitted result

\[
\frac{1}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \ldots}}}} = \prod_{n=0}^{\infty} \left(1 - q^{5n+5} \right) \left(1 - q^{5n+1} \right) \left(1 - q^{5n+3} \right)
\]

\[
\prod_{n=0}^{\infty} \left(1 - q^{5n+2} \right) \left(1 - q^{5n+4} \right)
\]

which states that the fraction of the two left-hand sides (or the right-hand sides for that matter) in the Rogers-Ramanujan identities can be expressed as a continued fraction. This result together with proof can also be found in Rogers’ article from 1894 (see [9], p. 329).

\[\Box\]

### 3.3 Proof by Schur

As we have seen in the previous two subsections the two proofs of Theorem 3 by Rogers and Ramanujan are essentially the same. This is nothing but expected considering they were both highly inspired by Rogers’ first proof from 1894, where both the identities appear as corollaries to other results. I. Schur on the other hand seems not to have been influenced by the work of Rogers. This is vastly noticeable in the two proofs of the identities he contributed with (see [10], pp. 307-317).

Schur’s purely combinatorial proof is presented below and as we will see it fully differs from the two proofs by Rogers and Ramanujan. Because both of Schur’s proofs are different and because they were published around the same time as Rogers’ and Ramanujan’s he is said to also have rediscovered the identities.

**Proof of Theorem 3.** The number of partition of the form \( N = \sum_{j=1}^{M} \alpha_j \) will be denoted by \( B_1(N) \) when \( \alpha_j - \alpha_{j+1} \geq 2 \) and \( \alpha_j \geq 1 \), and \( B_2(N) \) when \( \alpha_j - \alpha_{j+1} \geq 2 \) and \( \alpha_j > 1 \). For \( n = 0 \) we define \( B_1(0) = B_2(0) = 1 \) and

\[
\phi_{\mu}(q) = \sum_{N=0}^{\infty} B_{\mu}(N) q^N,
\]

where \( \mu = 1, 2 \).

We wish to show

\[
\phi_1(q) = \prod_{n=0}^{\infty} \frac{1}{\left(1 - q^{5n+1} \right) \left(1 - q^{5n+4} \right)}, \quad (26)
\]

and

\[
\phi_2(q) = \prod_{n=0}^{\infty} \frac{1}{\left(1 - q^{5n+2} \right) \left(1 - q^{5n+3} \right)}, \quad (27)
\]
By multiplying both sides in (26) and (27) by \( \prod_{j=1}^{\infty} (1 - q^j) \) we obtain

\[
\prod_{j=1}^{\infty} (1 - q^j) \phi_1(q) = \prod_{n=0}^{\infty} (1 - q^{5(n+1)})(1 - q^{5n+2})(1 - q^{5n+3})
\]

and

\[
\prod_{j=1}^{\infty} (1 - q^j) \phi_2(q) = \prod_{n=0}^{\infty} (1 - q^{5(n+1)})(1 - q^{5n+1})(1 - q^{5n+4}).
\]

Using the Jacobi triple product we see

\[
\prod_{n=0}^{\infty} (1 - (q^2)^{2(n+1)})(1 - q^{-\frac{1}{2}} q^\frac{5}{2}(2n+1))(1 - q^{\frac{1}{2}} q^\frac{5}{2}(2n+1)) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{5n^2+n}{2}}
\]

and

\[
\prod_{n=0}^{\infty} (1 - (q^2)^{2(n+1)})(1 - q^{-\frac{3}{2}} q^\frac{5}{2}(2n+1))(1 - q^{\frac{3}{2}} q^\frac{5}{2}(2n+1)) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{5n^2+3n}{2}}.
\]

So proving (26) and (27) is equivalent to proving

\[
\prod_{j=1}^{\infty} (1 - q^j) \phi_\mu(q) = \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{5m^2 + (2\mu - 1)m}{2}},
\]  

(28)

where \( \mu = 1, 2 \). When the left hand side in (28) is expanded as a power series of the form \( \sum_{n=0}^{\infty} A_\mu(n) q^n \), the coefficients \( A_\mu(n) \) can be interpreted in the following way

\[
n = \sum_{i=1}^{\alpha} x_i + \sum_{i=1}^{\beta} y_i
\]

with

\[
x_i > x_{i+1}, \quad x_\alpha \geq 1, \quad y_i \geq y_{i+1} + 2, \quad y_\beta \geq \mu.
\]

(29)

By extending the sum over all such partitions we get \( A_\mu(n) = \sum (-1)^\alpha \).

In what follows a partition will be denoted by

\[
(X|Y) = (x_1, \ldots, x_\alpha | y_1, \ldots, y_\beta)
\]

and called even if \( \alpha \) is even and odd if \( \alpha \) is odd. Furthermore if \( x_i \) or \( y_i \) are absent from a partition we denote it by \((-|Y)\) or \((X|-)\).
In order to show (28) we will show that there is a one-to-one correspondence between the class of even and odd partitions except when \( n \) is of the form
\[
    n = \frac{5m^2 + (2\mu - 1)m}{2}, \quad \mu = 1, 2.
\]
In the above described case we will show that the class of partitions for which \( \alpha \equiv m \pmod{2} \) contains one more partitions than the class for which \( \alpha \not\equiv m \pmod{2} \). This will then lead to \( A_{\mu}(\frac{5m^2 + (2\mu - 1)m}{2}) = (-1)^m \) and 0 otherwise.

We begin the construction of the one-to-one correspondence by pairing \((x_1, \ldots, x_\alpha | y_1, \ldots, y_\beta)\) with \((x_2, \ldots, x_\alpha | x_1, y_1, \ldots, y_\beta)\) if both satisfy (29).

After this matching we are left with partitions satisfying (29) but without a partner of the described form. These leftovers are then placed into two sets \( \Lambda_1 \) and \( \Lambda_2 \), where \( \Lambda_1 \) is the set for which \( x_1 = y_1 \) and \( \Lambda_2 \) is the set for which \( x_1 = y_1 + 1 \geq 2 \). We then introduce three characteristic numbers \( p, r \) and \( s \).

Here \( p = 0 \) if \( \alpha = 0 \) and \( p = x_\alpha \) if \( \alpha > 0 \). The number \( r \) is defined as the largest positive integer such that \( x_1 - x_2 = x_2 - x_3 = \ldots = x_{r-1} - x_r = 1 \) and \( s \) is defined as the largest positive integer such that \( y_1 - y_2 = y_2 - y_3 = \ldots = y_{s-1} - y_s = 2 \). Defining \( m = \min(p, r, s) \) and dividing \( \Lambda_1 \) and \( \Lambda_2 \) into three classes according to which one of the characteristic numbers attains this minimum we get the following six subsets

\[ \Lambda_1^{(1)} : \quad p = m, \quad r \geq m, \quad s \geq m, \]
\[ \Lambda_1^{(2)} : \quad p > m, \quad r \geq m, \quad s = m, \]
\[ \Lambda_1^{(3)} : \quad p > m, \quad r = m, \quad s > m, \]

and

\[ \Lambda_2^{(1)} : \quad p > m, \quad r = m, \quad s \geq m, \]
\[ \Lambda_2^{(2)} : \quad p = m, \quad r \geq m, \quad s \geq m, \]
\[ \Lambda_2^{(3)} : \quad p > m, \quad r > m, \quad s = m. \]

We continue by defining three maps \( f_1, f_2, f_3 \) and their inverses \( f_1^{-1}, f_2^{-1}, f_3^{-1} \).

If
\[ (X|Y) = (x_1, \ldots, x_\alpha | y_1, \ldots, y_\beta) \in \Lambda_1^{(1)} \]
we set
\[ f_1(X|Y) = (x_1 + 1, \ldots, x_m + 1, x_{m+1}, \ldots, x_{\alpha-1} | y_1, \ldots, y_\beta). \]

Clearly \( f_1(X|Y) \in \Lambda_2^{(1)} \) except when \( \alpha = m \). Then since \( p = m, r \geq m \) and \( x_1 = y_1 \) we have
\[ (X|Y) = (2m - 1, 2m - 2, \ldots, m | 2m - 1, 2m - 3, \ldots, 1). \]
So
\[ n = \sum_{j=m}^{2m-1} j + \sum_{j=0}^{m-1} (2j + 1) = \frac{3m - 1}{2} m + \frac{2m}{2} m = \frac{5m^2 - m}{2}. \]

Conversely, if
\[ (X|Y) = (x_1, \ldots, x_{\alpha}|y_1, \ldots, y_{\beta}) \in \Lambda_2^{(1)} \]
we set
\[ f_1^{-1}(X|Y) = (x_1 - 1, \ldots, x_m - 1, x_{m+1}, \ldots, x_{\alpha-1}, m|y_1, \ldots, y_{\beta}). \]

Obviously \( f_1^{-1}(X|Y) \in \Lambda_1^{(1)} \) except for \( \alpha = m \) and \( x_m - 1 = m \). In that case, since \( r = m, s \geq m \) and \( x_1 = b_1 - 1 \), we have
\[ (X|Y) = (2m, 2m - 1, \ldots, m + 1|2m - 1, 2m - 3, \ldots, 1). \]

So
\[ n = \sum_{j=m+1}^{2m} j + \sum_{j=0}^{m-1} (2j + 1) = \frac{3m + 1}{2} m + \frac{2m}{2} m = \frac{5m^2 + m}{2}. \]

Moreover if
\[ (X|Y) = (x_1, \ldots, x_{\alpha}|y_1, \ldots, y_{\beta}) \in \Lambda_1^{(2)} \]
we set
\[ f_2(X|Y) = (x_1, \ldots, x_{\alpha}, m|y_1 - 1, \ldots, y_m - 1, y_{m+1}, \ldots, y_{\beta}). \]

We then have \( f_2(X|Y) \in \Lambda_2^{(2)} \) except when \( \mu = 2 \) and \( y_{\beta} = 2 \). Since \( s = m, p > m, r \geq m \) and \( x_1 = y_1 \) we have
\[ (X|Y) = (2m, 2m - 1, \ldots, m + 1|2m, 2m - 2, \ldots, 2). \]

So
\[ n = \sum_{j=m+1}^{2m} j + \sum_{j=1}^{m} 2j = \frac{3m + 1}{2} m + \frac{2m + 2}{2} m = \frac{5m^2 + 3m}{2}. \]

If
\[ (X|Y) = (x_1, \ldots, x_{\alpha}|y_1, \ldots, y_{\beta}) \in \Lambda_2^{(2)} \]
we define the inverse \( f_2^{-1} \) by
\[ f_2^{-1}(X|Y) = (x_1, \ldots, x_{\alpha-1}|y_1 + 1, \ldots, y_m + 1, y_{m+1}, \ldots, y_{\beta}). \]

We then have \( f_2^{-1}(X|Y) \in \Lambda_1^{(2)} \).

Finally for
\[ (X|Y) = (x_1, \ldots, x_{\alpha}|y_1, \ldots, y_{\beta}) \in \Lambda_1^{(3)} \]
we define
\[ f_3(X|Y) = (y_1, x_1-1, \ldots, x_m-1, x_{m+1}, \ldots, x_\alpha|y_2+1, \ldots, y_{m+1}+1, y_{m+2}, \ldots, y_\beta). \]

We then have \( f_3(X|Y) \in \Lambda_2^{(3)} \). Conversely for
\[ (X|Y) = (x_1, \ldots, x_\alpha|y_1, \ldots, y_\beta) \in \Lambda_2^{(3)} \]
we set
\[ f_3^{-1}(X|Y) = (x_2+1, \ldots, x_{m+1}+1, x_{m+2}, \ldots, x_\alpha|x_1, y_1-1, \ldots \]
\[ \ldots, y_m-1, y_{m+1}, \ldots, y_\beta). \]

Now \( f_3^{-1}(X|Y) \in \Lambda_1^{(3)} \) except when \( \mu = 2 \) and \( y_\beta = 2 \). Since \( s = m, p > m, r > m \) and \( x_1 = y_1 + 1 \) in that case, we have
\[ (X|Y) = (2m+1, 2m, \ldots, m+1|2m, 2m-2, \ldots, 2). \]

And hence
\[ n = \sum_{j=m+1}^{2m+1} j + \sum_{j=1}^m 2j = \frac{3m+2}{2}(m+1) + \frac{2m+2}{2}m = \frac{5(m+1)^2-3(m+1)}{2}. \]

So we have a one-to-one correspondence between the class of even and odd partitions except when \( n = \frac{5m^2+(2\mu-1)n}{2} \). Furthermore \( \alpha \equiv m \pmod{2} \) for the spare partition \((x_1, \ldots, x_\alpha|y_1, \ldots, y_\beta)\). Thus (28) holds and the theorem follows.

■
4 Generalizations of The Rogers-Ramanujan Identities

In this section our main focus is to show that the following generalization of the Rogers-Ramanujan identities holds for all moduli. The below stated theorem first appeared in [6].

**Theorem 4** Let \( i = 0 \) or \( 1 \). The number of partitions of \( N \) with parts not congruent to \( 0, \pm d \mod (2k+i) \), where \( 1 \leq d < (2k+i)/2 \), is equal to the number of partitions of the form \( N = \sum_{j=1}^{\infty} \alpha_j \) where \( \alpha_j \geq \alpha_{j+1} \), \( \alpha_j \geq \alpha_{j+k-1} + 2 \), if \( \alpha_j \leq \alpha_{j+k-2} + 1 \) then \( \alpha_j + \alpha_{j+1} + \ldots + \alpha_{j+k-2} \equiv d - 1 \pmod{2-i} \) and at most \( d-1 \) of the \( \alpha_j = 1 \).

**Remark** We see that for \( i = 1 \), \( k = 2 \) and \( d = 1 \) or \( 2 \) we have the Rogers-Ramanujan identities. Furthermore we see that \( i = 0 \), \( k = 3 \) and \( d = 2 \) yields Euler’s classical result, i.e. that the number of partitions into odd parts is equal to the number of partitions into distinct parts. ▲

In order to show that Theorem 4 holds for all the cases, and not just the ones mentioned in the remark above, we will use results due to Gordon (see [7]), Andrews (see [5]) and Bressoud (see [6]). For the case \( i = 1 \) and \( k, d \) arbitrary Gordon’s generalization will be used. The proof to this particular generalization is actually a generalized version of Schur’s proof presented in Subsection 3.3. For the case \( i = 0 \), \( k \) odd and \( d \) arbitrary we will use results due to Andrews. Finally for the remaining case, i.e. for \( i = 0 \), \( k \) even and \( d \) arbitrary we will use Bressoud’s generalization of the identities. As we will see, the proof of the latter one is very much inspired by Andrews’ proofs.

4.1 Generalization by Gordon

**Theorem 4.1** The number of partitions of \( N \) with parts not congruent to \( 0, \pm d \mod (2k+1) \), where \( 1 \leq d \leq k \), is equal to the number of partitions of the form \( N = \sum_{j=1}^{M} \alpha_j \) where \( \alpha_j \geq \alpha_{j+1} \), \( \alpha_j \geq \alpha_{j+k-1} + 2 \) and \( \alpha_{M-d} \geq 2 \).

**Proof.** Denote by \( B_{k,d}(N) \) the number of partitions of the form \( N = \alpha_1 + \ldots + \alpha_M \) satisfying the conditions stated in the theorem. Furthermore let \( B_{k,d}(0) = 1 \) and define

\[
\phi_{k,d}(q) = \sum_{N=0}^{\infty} B_{k,d}(N)q^N.
\]

We wish to show

\[
\phi_{k,d}(q) = \prod_{n \neq 0, \pm d \mod (2k+1)} (1 - q^n)^{-1}.
\]
By Jacobi’s triple product we have
\[
\prod_{n=0}^{\infty} (1 - q^{(2k+1)(n+1)})(1 - q^{(2k+1)(n+1)-d})(1 - q^{(2k+1)n+d}) = 
\prod_{n=0}^{\infty} (1 - q^{(k+\frac{1}{2})(2(n+1))})(1 - q^{-k+d+\frac{1}{2}q(k+\frac{1}{2})(2(n+1))})(1 - q^{-k+d-\frac{1}{2}q(k+\frac{1}{2})(2(n+1))})
= \sum_{n=-\infty}^{\infty} (-q^{k-d+\frac{1}{2}})^n (q^{k+\frac{1}{2}})^n^2 = \sum_{n=-\infty}^{\infty} (-1)^n q^{(k+\frac{1}{2}) n^2 + (k-d+\frac{1}{2}) n}.
\]

Therefore, in order to prove (30) it is sufficient to prove the following
\[
\prod_{j=1}^{\infty} (1 - q^{j}) \varphi_{k,d}(q) = \sum_{m=-\infty}^{\infty} (-1)^m q^{(k+\frac{1}{2}) m^2 + (k-d+\frac{1}{2}) m}.
\] (31)

The left hand side in (31) can be expanded as a power series of the following form
\[
\sum_{n=0}^{\infty} A_{k,d}(n) q^n,
\]
where the coefficients $A_{k,d}(n)$ can be interpreted in the following way: if
\[
n = \sum_{i=1}^{\alpha} x_i + \sum_{i=1}^{\beta} y_i
\]
we have
\[
x_i > x_{i+1}, y_i \geq y_{i+1}, y_i \geq y_{i+k-1} + 2, y_{\beta+d+1} \geq 2.
\] (32)

When extending the sum over all such partitions we have $A_{k,d}(n) = \sum (-1)^n$. In the sequel we will denote a partition by
\[
(X|Y) = (x_1, \ldots, x_\alpha|y_1, \ldots, y_\beta),
\]
Furthermore, we call the partitions satisfying (32) even if $\alpha$ is even and odd if $\alpha$ is odd.

We will show that there is a one-to-one correspondence between the class of even and odd partitions except when $n$ has the following form
\[
n = (k + \frac{1}{2}) m^2 + (k - d + \frac{1}{2}) m.
\]
In the latter case we will show that the class of partitions for which $\alpha \equiv m \pmod{2}$ contains one more partition than the class of partition for which $\alpha \not\equiv m \pmod{2}$.
As a first step in setting up the one-to-one correspondence between even and odd partitions we do as follows: if both \((x_1, \ldots, x_\alpha | y_1, \ldots, y_\beta)\) and \((x_2, \ldots, x_\alpha | x_1, y_1, \ldots, y_\beta)\) satisfy (32) we pair them with each other,

\[(x_1, \ldots, x_\alpha | y_1, \ldots, y_\beta) \leftrightarrow (x_2, \ldots, x_\alpha | x_1, y_1, \ldots, y_\beta).
\]

But not all partitions satisfying (32) have a partner of the form described above. So we are left with a set \(\Lambda\) of partitions \((x_1, \ldots, x_\alpha | y_1, \ldots, y_\beta)\) satisfying (32) such that \(y_1 \leq x_1 \leq y_1 + 1\), i.e. neither \((x_2, \ldots, x_\alpha | x_1, y_1, \ldots, y_\beta)\) nor \((y_1, x_1, \ldots, x_\alpha | y_2, \ldots, y_\beta)\) satisfies (32).

Next we split our \(\Lambda\) into subsets \(\Lambda_i\) \((1 \leq i \leq k)\) with partitions satisfying

\[x_1 = y_1 = \ldots = y_{i-1} = y_i + 1 = \ldots = y_{k-1} + 1.
\]

These subsets are clearly disjoint, and in \(\Lambda_1\) we have \(x_1 = y_1 + 1 = \ldots = y_{k-1} + 1\) while in \(\Lambda_k\) we have \(x_1 = y_1 = \ldots = y_{k-1}\).

We continue by introducing four characteristic numbers \(p, r, s\) and \(t\) for the partitions \((x_1, \ldots, x_\alpha | y_1, \ldots, y_\beta)\) in the following way: \(p = x_\alpha\), \(r\) is the largest number such that \(x_1 - x_2 = \ldots = x_r - x_1 = 1\), \(s\) is the largest number such that \(y_k - y_{2(k-1)} = \ldots = y_{(s-1)(k-1)} - y_{s(k-1)} = 2\) and \(t\) is the largest number such that \(y_i - y_{i+1} - i = \ldots = y_{i-1} + (t-2)(k-1) - y_{i-1+(t-1)(k-1)} = 2\), where \(t\) only exists for \(2 \leq i \leq k-1\).

Let \(m = \min(p, r, s)\) and subdivide \(\Lambda_1\) and \(\Lambda_k\) into three classes and \(\Lambda_2, \ldots, \Lambda_{k-1}\) into four classes as follows

\[
\begin{align*}
\Lambda_1^{(1)} : & \quad p = m, \ r \geq m, \ s \geq m, \\
\Lambda_1^{(2)} : & \quad p > m, \ r = m, \ s \geq m, \\
\Lambda_1^{(3)} : & \quad p > m, \ r > m, \ s = m, \\
\Lambda_k^{(1)} : & \quad p = m, \ r \geq m, \ s \geq m, \\
\Lambda_k^{(2)} : & \quad p > m, \ r \geq m, \ s = m, \\
\Lambda_k^{(4)} : & \quad p > m, \ r = m, \ s > m, \\
\Lambda_i^{(1)} : & \quad p = m, \ r \geq m, \ s \geq m, \\
\Lambda_i^{(2)} : & \quad p > m, \ r \geq m, \ s \geq m, \ t = m, \\
\Lambda_i^{(3)} : & \quad p > m, \ r = m, \ s \geq m, \ t > m, \\
\Lambda_i^{(4)} : & \quad p > m, \ r > m, \ s = m, \ t > m,
\end{align*}
\]

and continue by defining three maps \(F_1, F_2, F_3\) and their inverses \(F_1^{-1}, F_2^{-1}, F_3^{-1}\).
If

\( (X|Y) = (x_1, \ldots, x_\alpha | y_1, \ldots, y_\beta) \in \Lambda_1^{(2)}, \)

we set

\[ F_1(X|Y) = (x_1 - 1, \ldots, x_m - 1, x_{m+1}, \ldots, x_\alpha | y_1, \ldots, y_\beta). \]

Clearly \( F_1(X|Y) \in \Lambda_1^{(1)} \) except when \( \alpha = m \) and \( x_m - 1 = m \). In that case, since \( r = m \) and \( s \geq m \), we have

\( (X|Y) = (2m, 2m - 1, \ldots, m + 1 | 2m - 1, \ldots, 2m - 1, \overbrace{2m - 3, \ldots, 2m - 3}^{k-1}, \overbrace{1, \ldots, 1}^{k-1}). \)

So

\[ n = \sum_{j=0}^{m-1} (2m - j) + (k - 1) \sum_{j=0}^{m-1} (2m - (2j + 1)) = \frac{3m^2 + m}{2} + (k - 1) \frac{2m^2}{2} = \left( \frac{3 + 2(k - 1)}{2} \right) m^2 + \frac{m}{2} = (k + \frac{1}{2}) m^2 + \frac{m}{2}. \]

Conversely if

\( (X|Y) = (x_1, \ldots, x_\alpha | y_1, \ldots, y_\beta) \in \Lambda_1^{(1)}, \)

we set

\[ F_1^{-1}(X|Y) = (x_1 + 1, \ldots, x_m + 1, x_{m+1}, \ldots, x_{\alpha-1} | y_1, \ldots, y_\beta). \]

This now gives us \( F_1^{-1}(X|Y) \in \Lambda_1^{(2)} \) except when \( \alpha = m \). In that case, since \( p = m, r \geq m \) and \( s \geq m \), we have

\( (X|Y) = (2m - 1, 2m - 2, \ldots, m | 2m - 1, \ldots, 2m - 1, \overbrace{2m - 3, \ldots, 2m - 3}^{k-1}, \overbrace{1, \ldots, 1}^{k-1}). \)

So

\[ n = \sum_{j=1}^{m} (2m - j) + (k - 1) \sum_{j=0}^{m-1} (2m - (2j + 1)) = \frac{3m^2 - m}{2} + (k - 1) \frac{2m^2}{2} = (k + \frac{1}{2}) m^2 - \frac{m}{2}. \]

Now if \( 1 \leq i \leq k - 1 \) we have that

\( (X|Y) = (x_1, \ldots, x_\alpha | y_1, \ldots, y_\beta) \in \Lambda_i^{(1)}, \)
we set
\[ F_2(X|Y) = (x_1, \ldots, x_{\alpha-1}|y_1, \ldots, y_{i-1}, y_i + 1, y_{i+1}, \ldots, y_{i+(k-1)}, y_{i+(k-1)} + 1, \ldots, y_{i+(m-1)(k-1)} + 1, \ldots). \]

We then have \( F_2(X|Y) \in \Lambda_{i+1}^{(2)} \). Conversely if
\[ (X|Y) = (x_1, \ldots, x_{\alpha}|y_1, \ldots, y_{\beta}) \in \Lambda_{i+1}^{(2)}, \]
we set
\[ F_{2}^{-1}(X|Y) = (x_1, \ldots, x_{\alpha}, m|y_1, \ldots, y_{i-1}, y_i - 1, y_{i+1}, \ldots, y_{i+(k-1)}, y_{i+(k-1)} - 1, \ldots, y_{i+(m-1)(k-1)} - 1, \ldots). \]

This then gives us \( F_{2}^{-1}(X|Y) \in \Lambda^{(1)}_i \). But we note that \( F_{2}^{-1}(X|Y) \) violates the condition \( y_{\beta-d+1} \geq 2 \) if and only if \( i = k - d \) and
\[
(X|Y) = (2m, 2m - 1, \ldots, m + 1|2m, \ldots, 2m - 1, 2m - 1, \ldots, 2m - 1, 2m - 2, \ldots, 2m - 2, \ldots, 1, \ldots, 1).
\]

So
\[
n = \sum_{j=0}^{m-1} (2m - j) + (k - d) \sum_{j=0}^{m-1} (2m - 2j) + (d - 1) \sum_{j=0}^{m-1} (2m - (2j + 1))
= \frac{3m^2 + m}{2} + (k - d) \frac{2m^2 + 2m}{2} + (d - 1) \frac{2m^2}{2}
= \left( \frac{3 + 2(k - d) + 2(d - 1)}{2} \right) m^2 + \left( \frac{1 + 2(k - d)}{2} \right) m
= \left( k + \frac{1}{2} \right) m^2 + (k - d + \frac{1}{2}) m.
\]

Finally if
\[ (X|Y) = (x_1, \ldots, x_{\alpha}|y_1, \ldots, y_{\beta}) \in \Lambda^{(4)}_i, \]
we set
\[ F_3(X|Y) = (y_1, x_1 - 1, \ldots, x_{m-1}, x_m + 1, \ldots, x_{\alpha}|y_2, \ldots, y_{k-1}, y_k + 1, \]
\[ y_{k+1}, \ldots, y_{k+(k-1)} + 1, \ldots, y_{k+(m-1)(k-1)} + 1, \ldots). \]

Obviously \( F_3(X|Y) \in \Lambda^{(3)}_{i-1} \). Conversely if
\[ (X|Y) = (x_1, \ldots, x_{\alpha}|y_1, \ldots, y_{\beta}) \in \Lambda^{(3)}_{i-1}, \]
we set
\[ F^{-1}_3(X|Y) = (x_2 + 1, \ldots, x_{m+1} + 1, x_{m+2}, \ldots, x_{\alpha}|x_1, y_1, \ldots, y_{k-1} - 1, \ldots, y_{(m-1)(k-1) - 1}, \ldots). \]

This then leads to \( F^{-1}_3(X|Y) \in \Lambda_i^{(4)} \). However \( F^{-1}_3(X|Y) \) violates the condition \( y_{\beta-d+1} \geq 2 \) if and only if \( i = t \) and
\[
(X|Y) = (2m - 1, 2m - 2, \ldots, m) \overbrace{2m - 1, \ldots, 2m - 1}^{d-1}, \overbrace{2m - 2, \ldots, 2m - 2}^{k-d}, \overbrace{2m - 3, \ldots, 2m - 3}^{d-1}, \overbrace{1, \ldots, 1}^{d-1}.
\]

So
\[
n = \sum_{j=1}^{m} (2m - j) + (d - 1) \sum_{j=0}^{m-1} (2m - (2j + 1)) + (k - d) \sum_{j=1}^{m-1} (2m - 2j)
= \frac{3m^2 - m}{2} + (d - 1) \frac{2m^2}{2} + (k - d) \frac{2m^2 - 2m}{2}
= \left( \frac{3 + 2(d - 1) + 2(k - d)}{2} \right) m^2 + \left( -1 - 2(k - d) \right) m
= (k + \frac{1}{2})m^2 - (k - d + \frac{1}{2})m.
\]

We now have a one-to-one correspondence between the class of even partitions and the class of odd partitions, except when \( n = (k + \frac{1}{2})m^2 + (k - d + \frac{1}{2})m, m \in \mathbb{Z} \). We also know that for the spare partition in that case \( \alpha \equiv m \) (mod 2). So we have
\[
A_{k,d}(n) = \begin{cases} 
(-1)^m & \text{if } n = (k + \frac{1}{2})m^2 + (k - d + \frac{1}{2})m, \\
0 & \text{otherwise}.
\end{cases}
\]

Thus (31) holds and the theorem follows.

\section*{4.2 Generalization by Andrews}

\textbf{Theorem 4.2} The number of partitions of \( N \) with parts not congruent to 0, \( \pm(2d + 1) \) (mod \( 4k + 2 \)), where \( 0 \leq d < k \), is equal to the number of partitions of the form \( N = \sum_{j=1}^{\infty} \beta_j \cdot j \) where \( \beta_1 \leq 2d \) and \( \beta_{j+1} \leq 2(k-1)-\beta_j \).

Define
\[
c_{k,i}(x,q) = \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{\frac{1}{2}(2k+1)n(n+1)-in} (1 - x^i q^{(2n+1)i}) \frac{(xq;q)_n}{(q;q)_n}. \tag{33}
\]
Before continuing with the proof of Theorem 4.2 we are going to need the below stated lemma. According to Andrews (see [4], p. 424) the results contained in the lemma were first proved by Selberg. Unfamiliar with Selberg’s proof we give a proof inspired by Andrews’ proof of Lemma 7.1 in [2] p. 106.

**Lemma 4.2.1** For $c_{k,i}(x, q)$ as in (33) we have

(i) $c_{k,-i}(x, q) = -x^{-i}q^{-i}c_{k,i}(x, q)$,

(ii) $c_{k,i}(x, q) - c_{k,i-1}(x, q) = x^{i-1}q^{i-1}(1 - xq)c_{k,k-i+1}(xq, q)$.

**Proof.** For identity (i) we have

\[
c_{k,-i}(x, q) = \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{\frac{1}{2}(2k+1)n(n+1)+in \left(1 - x^{-i}q^{-(2n+1)i}\right)} \frac{(xq; q)_n}{(q; q)_n}
\]

\[
= \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{\frac{1}{2}(2k+1)n(n+1)+in \left(1 - x^{-i}q^{-(2n+1)i}\right)}(x^i q^{2n+1}) - 1 \frac{(xq; q)_n}{(q; q)_n}
\]

\[
= -x^{-i}q^{-i} \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{\frac{1}{2}(2k+1)n(n+1)-in \left(1 - x^i q^{2n+1}\right)} \frac{(xq; q)_n}{(q; q)_n}
\]

\[
= -x^{-i}q^{-i}c_{k,i}(x, q).
\]

For identity (ii) we first notice that

\[
q^{-in}(1 - x^i q^{2n+1}) + q^{-(i-1)n}(1 - x^{-i}q^{2n+1)(i-1)})
\]

\[
= q^{-in}(1 - q^n) + x^{i-1}q^{(i-1)(n+1)}(1 - q^{n+1}).
\]

By using the above the left-hand side of the identity can be rewritten as follows

\[
c_{k,i}(x, q) - c_{k,i-1}(x, q) =
\]

\[
= \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{\frac{1}{2}(2k+1)n(n+1)} q^{-in \left(1 - x^i q^{2n+1}\right)} \frac{(xq; q)_n}{(q; q)_n}
\]

\[
+ \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{\frac{1}{2}(2k+1)n(n+1)} q^{-(i-1)n \left(1 - x^{-i} q^{2n+1)(i-1)}\right)} \frac{(xq; q)_n}{(q; q)_n}
\]

\[
= \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{\frac{1}{2}(2k+1)n(n+1)} q^{-in} \frac{(xq; q)_n}{(q; q)_n}
\]

\[
+ \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{\frac{1}{2}(2k+1)n(n+1)} x^{-i-1} q^{-(i-1)(n+1)} \frac{(xq; q)_n}{(q; q)_n}
\]

\[
= \sum_{n=1}^{\infty} (-1)^n x^{kn} q^{\frac{1}{2}(2k+1)n(n+1)} q^{-in} \frac{(xq; q)_n}{(q; q)_{n-1}}
\]

\[
+ \sum_{n=0}^{\infty} (-1)^n x^{kn} q^{\frac{1}{2}(2k+1)n(n+1)} x^{-i-1} q^{-(i-1)(n+1)} \frac{(xq; q)_{n+1}}{(q; q)_n}
\]

\[
= 30.
\]
By the second identity of Lemma 4.2.1 and (34) we have the following for \( 0 \leq i \leq k \):

\[
R_{k,i}(x, q) - R_{k,i-1}(x, q) = (c_{k,i+\frac{1}{2}}(x^2, q^2) - c_{k,i-\frac{1}{2}}(x^2, q^2)) \prod_{j=1}^{\infty} (1 - xq^j)^{-1}
\]

\[
= (x^2)^{i+\frac{1}{2}-1}(q^2)^{i+\frac{1}{2}-1}(1 - x^2q^2) c_{k,k-(i+\frac{1}{2})+1}(x^2, q^2) \prod_{j=1}^{\infty} (1 - xq^j)^{-1}
\]

which proves identity (ii).

Proof of Theorem 4.2. Denote by \( A_{k,d}(N) \) the number of partitions of the integer \( N \) into parts \( \neq 0, \pm(2d + 1) \mod 4k + 2 \) and by \( B_{k,d}(N) \) the number of partitions of \( N \) of the form \( \sum_{j=1}^{\infty} \beta_j \cdot j \) satisfying the two conditions stated in the theorem. We wish to show \( A_{k,d}(N) = B_{k,d}(N) \).

We will begin by discussing the second condition on the partitions enumerated \( B_{k,d}(N) \). By that condition we know that if \( \beta_j = 2j - 1 \) or \( 2j \) we have \( \beta_{j+1} \leq 2(k - 1) - \beta_j \leq 2(k - j) \). By setting \( j = k \) we can therefore conclude that no part of \( N \) appears more that 2 times.

Define

\[
R_{k,i}(x, q) = c_{k,i+\frac{1}{2}}(x^2, q^2) \prod_{n=1}^{\infty} (1 - xq^n)^{-1}.
\]  

(34)

By the second identity of Lemma 4.2.1 and (34) we have the following for \( 0 \leq i \leq k \):

\[
R_{k,i}(x, q) - R_{k,i-1}(x, q) = (c_{k,i+\frac{1}{2}}(x^2, q^2) - c_{k,i-\frac{1}{2}}(x^2, q^2)) \prod_{j=1}^{\infty} (1 - xq^j)^{-1}
\]

\[
= (x^2)^{i+\frac{1}{2}-1}(q^2)^{i+\frac{1}{2}-1}(1 - x^2q^2) c_{k,k-(i+\frac{1}{2})+1}(x^2, q^2) \prod_{j=1}^{\infty} (1 - xq^j)^{-1}
\]

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\[
\begin{align*}
&= x^{2i-1}q^{2i-1}(1 + xq)c_{k,k-i+\frac{1}{2}}(x^2q^2,q^2) \prod_{j=1}^{\infty}(1 - xq^{j+1})^{-1} \\
&= x^{2i-1}q^{2i-1}(1 + xq)R_{k,k-i}(xq,q).
\end{align*}
\]

And hence

\[
R_{k,i}(x,q) - R_{k,i-1}(x,q) = x^{2i}q^{2i}(1 + x^{-1}q^{-1})R_{k,k-i}(xq,q). \tag{35}
\]

By the first identity of Lemma 4.2.1 we have

\[
R_{k,-1}(x,q) = c_{k,-\frac{1}{2}}(x^2,q^2) \prod_{j=1}^{\infty}(1 - xq^j)^{-1} \\
= -(x^2)^{-\frac{1}{2}}(q^2)^{-\frac{1}{2}}c_{k,\frac{1}{2}}(x^2,q^2) \prod_{j=1}^{\infty}(1 - xq^j)^{-1} \\
= -x^{-1}q^{-1}R_{k,0}(x,q). \tag{36}
\]

Using (35) and (36) it follows that

\[
(1 + x^{-1}q^{-1})R_{k,k}(xq,q) = R_{k,0}(x,q) - R_{k,-1}(x,q) \\
= R_{k,0}(x,q) + x^{-1}q^{-1}R_{k,0}(x,q) \\
= (1 + x^{-1}q^{-1})R_{k,0}(x,q).
\]

By dividing both sides by \((1 + x^{-1}q^{-1})\) we get

\[
R_{k,0}(x,q) = R_{k,k}(xq,q). \tag{37}
\]

For \(|x| \leq 1\) and \(|q| < 1\) we can express \(R_{k,i}(x,q)\) in the following way

\[
R_{k,i}(x,q) = \sum_{N=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} a_{k,i}(M,N)x^Mq^N. \tag{38}
\]

Then by (34), (35), (37) and (38) we have

\[
a_{k,i}(M,N) = \begin{cases} 
1 & \text{if } M = N = 0 \\
0 & \text{if either } M \leq 0 \text{ or } N \leq 0 \text{ and } (M,N) \neq (0,0), 
\end{cases} \tag{39}
\]

and

\[
a_{k,0}(M,N) = a_{k,k}(M,N-M), \tag{40}
\]

and

\[
a_{k,i}(M,N) - a_{k,i-1}(M,N) = \\
= a_{k,k-i}(M-2i+1,N-M) + a_{k,k-i}(M-2i,N-M). \tag{41}
\]

where \(a_{k,i}(M,N)\) is uniquely determined by (39), (40) and (41).
Now let $b_{k,i}(M, N)$ denote the number of partitions of $N$ into $M$ parts of the form $\sum_{j=1}^{\infty} \beta_j \cdot j$, where $\beta_j \geq 0$ denotes the number of times the summand $j$ appears in the partition, with $\beta_1 \leq 2i$ and $\beta_{j+1} \leq 2(k-1) - \beta_j$. We wish to show $b_{k,i}(M, N) = a_{k,i}(M, N)$, i.e. we wish to show that $b_{k,i}(M, N)$ satisfies (39), (40) and (41).

For (40) consider any partition enumerated by $a_{k,0}(M, N)$. Since $\beta_1 \leq 2 \cdot 0 = 0$ we have

$$N = \beta_2 \cdot 2 + \beta_3 \cdot 3 + \beta_4 \cdot 4 + \ldots = \sum_{j=2}^{\infty} \beta_j \cdot j,$$

where $\sum_{j=2}^{\infty} \beta_j = M$. By subtracting 1 from every summand we obtain

$$\sum_{j=2}^{\infty} \beta_j (j-1) = \sum_{j=2}^{\infty} \beta_j \cdot j - \sum_{j=2}^{\infty} \beta_j = N - M,$$

where the left hand side can be rewritten in the following way

$$\sum_{j=2}^{\infty} \beta_j (j-1) = \sum_{j=1}^{\infty} \beta_{j+1} \cdot j = \sum_{j=1}^{\infty} \tilde{\beta}_j \cdot j.$$

We therefore obtain a partition of $N - M$ into $M$ parts with $\tilde{\beta}_1 = \beta_2 \leq 2k$ and $\tilde{\beta}_{j+1} \leq 2(k-1) - \beta_j$. Thus we have a partition enumerated by $b_{k,k}(M, N - M)$ and hence

$$b_{k,0}(M, N) = b_{k,k}(M, N - M).$$

For (41) we note that $b_{k,i}(M, N) - b_{k,i-1}(M, N)$ is the number of partitions of $N$ into $M$ parts of the form $\sum_{j=1}^{\infty} \beta_j \cdot j$, where $\beta_1 = 2i - 1$ or $2i$ and $\beta_{j+1} \leq 2(k-1) - \beta_j$. If $\beta_1 = 2i - 1$ it follows by the second condition that $\tilde{\beta}_2 \leq 2(k-1) - \beta_1 \leq 2(k-i)$. Again, by subtracting 1 from every summand we obtain

$$\sum_{j=1}^{\infty} \beta_j (j-1) = \sum_{j=1}^{\infty} \beta_j \cdot j - \sum_{j=1}^{\infty} \beta_j = N - M,$$

where the left hand side also can be expressed as

$$\sum_{j=1}^{\infty} \beta_j (j-1) = \sum_{j=1}^{\infty} \beta_{j+1} \cdot j = \sum_{j=1}^{\infty} \tilde{\beta}_j \cdot j,$$

with $\sum_{j=1}^{\infty} \tilde{\beta}_j = M - 2i + 1$. So we obtain a partition of $N - M$ into $M - 2j + 1$ parts with $\tilde{\beta}_1 = \beta_2 \leq 2(k - j)$ and $\tilde{\beta}_{j+1} \leq 2(k-1) - \tilde{\beta}_j$. Consequently our partition has been transformed into one enumerated by $b_{k,k-i}(M-2i+1, N-M)$. If $\beta_1 = 2i$ we again obtain $\tilde{\beta}_2 \leq 2(k-i)$. Repeating the above procedure we obtain a partition of $N - M$ into $M - \beta_1 = M - 2i$
parts with $\tilde{\beta}_1 = \beta_2 \leq 2(k - i)$ and $\tilde{\beta}_{j+1} \leq 2(k - 1) - \tilde{\beta}_j$. So we end up with a partition enumerated by $b_{k,k-i}(M - 2i, N - M)$.

Finally we have

$$b_{k,i}(M, N) - b_{k,i-1}(M, N) = b_{k,k-i}(M - 2i + 1, N - M) + b_{k,k-i}(M - 2i, N - M).$$

Now since $b_{k,i}(M, N)$ satisfies (39) by definition it follows from uniqueness of $a_{k,i}(M, N)$ that

$$a_{k,i}(M, N) = b_{k,i}(M, N).$$

(42)

Noting that $B_{k,d}(N) = \sum_{M=0}^{\infty} b_{k,d}(M, N)$ and using (38) and (39) we see

$$\sum_{N=0}^{\infty} B_{k,d}(N)q^N = \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} b_{k,d}(M, N)q^N = \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} a_{k,d}(M, N)q^N = R_{k,d}(1, q).$$

(43)

We can rewrite $R_{k,d}(1, q)$ by expressing $c_{k,d+\frac{1}{2}}(1, q^2)$ as a Laurent series in the following

$$c_{k,d+\frac{1}{2}}(1, q^2) =$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n(n+1)-(d+\frac{1}{2})n(1-q^{2n+1})(d+\frac{1}{2})}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n(n+1)-(d+1)n(1-q^{2n+1})(d+1)}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n(n+1)-(2d+1)n}$$

$$- \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n(n+1)-(2d+1)n} q^{(2n+1)(2d+1)}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n(n+1)-(2d+1)n}$$

$$- \sum_{n=1}^{\infty} (-1)^n q^{(2k+1)(n-1)n-(2d+1)(n-1)} q^{(2n-1)+1)(2d+1)}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n(n+1)-(2d+1)n} + \sum_{n=1}^{\infty} (-1)^n q^{(2k+1)(n-1)n+(2d+1)n}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n(n+1)-(2d+1)n} + \sum_{n=1}^{\infty} (-1)^n q^{(2k+1)n(n+1)-(2d+1)n}$$

$$= \sum_{n=\infty}^{\infty} (-1)^n q^{(2k+1)n(n+1)-(2d+1)n},$$

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So by (34), (43) and the Jacobi triple product we have

\[
\sum_{N=0}^{\infty} B_{k,d}(N) q^N = R_{k,d}(1, q) = \sum_{n=-\infty}^{\infty} (-q^{2(k-d)})^n (q^{2k+1})^n \prod_{n=1}^{\infty} (1 - q^n)^{-1}
\]

\[
= \prod_{n=1}^{\infty} \left(1 - q^{(2k+1)(2n)}(1 - q^{2(k-d)}) q^{(2k+1)(2n-1)}(1 - q^{-2(k-d)} q^{(2k+1)(2n-1)}) \right)^{(1 - q^n)}
\]

\[
= \prod_{n=1}^{\infty} \left(1 - q^{(4k+2)n} (1 - q^{(4k+2)n - (2d+1)}(1 - q^{(4k+2)(n-1) + (2d+1)}) \right)^{(1 - q^n)}
\]

\[
= \prod_{n=1}^{\infty} (1 - q^n)^{-1} = \sum_{N=0}^{\infty} A_{k,d}(N) q^N.
\]

By comparing the coefficient of \(q^N\) in the two series the result follows. ■

The following theorem may be proved by mimicking the proof of Theorem 4.2

**Theorem 4.2’** The number of partitions of \(N\) with parts not congruent to \(0, \pm 2d \pmod{2k+2}\), where \(0 \leq d < k\), is equal to the number of partitions of the form \(N = \sum_{j=1}^{\infty} \beta_j \cdot j\) where \(\beta_1 \leq 2d - 1\) and \(\beta_{j+1} \leq 2k - 1 - \beta_j\).

**Remark** By using a more general definition of \(c_{k,i}(x, q)\) we should be able to combine the two theorems and give a proof that covers them both. This method is actually used both by Andrews (in his proof of Gordon’s generalization, see [3]) and by Bressoud (see the following subsection or [6]), where the more generalized version of \(c_{k,i}(x, q)\) is denoted by \(J_{k,d}(a, x, q)\) in the latter. Bressoud’s usage of the method is quite natural considering his proof is highly inspired by Andrews’.

### 4.3 Generalization by Bressoud

**Theorem 4.3** The number of partitions of \(N\) with parts not congruent to \(0, \pm d \pmod{2k}\), where \(0 < d < k\), is equal to the number of partitions of the form \(N = \sum_{j=1}^{\infty} \alpha_j \cdot j\) where \(\alpha_j \geq \alpha_{j+1}, \alpha_j \geq \alpha_{j+k-1} + 2, \text{ if } \alpha_j \leq \alpha_{j+k-2} + 1\) then \(\alpha_j + \alpha_{j+1} + \ldots + \alpha_{j+k-2} \equiv d - 1 \pmod{2}\) and at most \(d - 1\) of the \(\alpha_j = 1\).

Let \(|q| < 1\) and let \(|x| < |q^{-1}|\). Define

\[
H_{k,d}(a, x, q) = \sum_{n=0}^{\infty} x^{kn} q^{kn^2 + n^2 - d n a^n (1 - x d q^{2nd}) (a x q^{n+1}; q)_\infty (a^{-1}; q)_n} (q; q)_n (x q^n; q)_\infty (44)
\]

and

\[
J_{k,d}(a, x, q) = H_{k,d}(a, x, q) - a x q H_{k,d-1}(a, x, q).
\]
Since \(a^n(a^{-1};q)_n = (a - 1)(a - q)\ldots(a - q^{n-1})\) we note that the above defined works for any \(a\), even \(a = 0\).

For the sake of clarity, we will state and prove four necessary lemmas before proceeding to the actual proof of Theorem 4.3.

**Lemma 4.3.1** For \(H_{k,d}(a, x, q)\) and \(J_{k,d}(a, x, q)\) as in (44) and (45) respectively we have

\[H_{k,d}(a, x, q) - H_{k,d-1}(a, x, q) = x^{d-1}J_{k,k-d+1}(a, x, q).\]

**Proof.** First we notice the following

\[q^{-dn}(1 - x^d q^{2nd}) - q^{-(d-1)n}(1 - x^{d-1} q^{2n(d-1)}) =
q^{-dn}(1 - q^n) + x^{d-1} q^{n(2d-1)}(1 - xq^n).\]

Using this we can then rewrite the left-hand side of Lemma 4.3.1 in the following way

\[
egin{align*}
H_{k,d}(a, x, q) - H_{k,d-1}(a, x, q) &= \sum_{n=0}^{\infty} x^{kn} q^{kn^2 + dn - d_n a^n (axq^{n+1};q)_\infty (a^{-1};q)_n (1 - q^n) \\
&+ \sum_{n=0}^{\infty} x^{kn} q^{kn^2 + dn - d_n a^n (axq^{n+1};q)_\infty (a^{-1};q)_n x^{d-1} q^{2nd - n} (1 - xq^n) \\
&= \sum_{n=1}^{\infty} x^{kn} q^{kn^2 + dn - d_n a^n (axq^{n+1};q)_\infty (a^{-1};q)_n} (q; q)_{n-1} (xq^n; q)_{\infty}
+ \sum_{n=0}^{\infty} x^{kn} q^{kn^2 + dn - d_n a^n (axq^{n+1};q)_\infty (a^{-1};q)_n x^{d-1} q^{2nd - n}
= \sum_{n=0}^{\infty} x^{kn} q^{kn^2 + dn - d_n a^n (axq^{n+1};q)_\infty (a^{-1};q)_n} (q; q)_{n} (xq^{n+1}; q)_{\infty}
+ \sum_{n=0}^{\infty} x^{kn} q^{kn^2 + dn - d_n a^n (axq^{n+1};q)_\infty (a^{-1};q)_n x^{d-1} q^{2nd - n}
= \sum_{n=0}^{\infty} x^{kn} q^{kn^2 + dn - d_n a^n (axq^{n+1};q)_\infty (a^{-1};q)_n} (q; q)_{n} (xq^{n+1}; q)_{\infty}
+ \sum_{n=0}^{\infty} x^{kn} q^{kn^2 + dn - d_n a^n (axq^{n+1};q)_\infty (a^{-1};q)_n x^{d-1} q^{2nd - n}(1 - q^n/a),
\end{align*}
\]
Lemma 4.3.2 which proves the lemma.

\[ J_{k,d} \text{ left-hand side of the above stated lemma in the following way} \]

Proof. By using the definition of \( J_{k,d} \) in (45) we can rewrite the left-hand side of the above stated lemma in the following way.

\[ J_{k,d}(a, x, q) - J_{k,d-1}(a, x, q) = (xq)^{d-1}(J_{k,k-d+1}(a, x, q) - aJ_{k,k-d+2}(a, x, q)). \]

Lemma 4.3.3 For \( J_{k,d}(a, x, q) \) as in (45) we have

\[ (-xq; q)_\infty(J_{k-1, \frac{d}{2}}(a, x^2, q^2) - J_{k-1, \frac{d-1}{2}}(a, x^2, q^2)) = \]

\[ = (1 + xq)(-xq^2; q)_\infty(xq)^{d-2}(J_{k-1, \frac{k-d+1}{2}}(a, x^2, q^2) - aJ_{k-1, \frac{k-d+3}{2}}(a, x^2, q^2)). \]

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Proof. By Lemma 4.3.2 we have

\[ (-xq; q)_{\infty} (J_{\frac{k-1}{2}, \frac{d}{2}}(a, x^2, q^2) - J_{\frac{k-1}{2}, \frac{d}{2} - 1}(a, x^2, q^2)) \]

\[ = (-xq; q)_{\infty} (x^2q^2 \sum_{n=0}^{\infty} q^{(k-1)n}q^{(k-1)n^2+2n-d}aJ_{\frac{k-1}{2}, \frac{k-d+1}{2}}(a, x^2q^2, q^2)) \]

Now, since \((-xq; q)_{\infty} = (1 + xq)(-xq^2; q)_{\infty}\) we have the result.

Lemma 4.3.4 For \(J_{k,d}(a, x, q)\) as in (45) we have

\[ (-q; q)_{\infty} J_{\frac{k-1}{2}, \frac{d}{2}}(0, 1, q^2) = \prod_{n \in \mathbb{N}_{0, \pm d \text{mod } 2k}} (1 - q^n)^{-1}. \]

Proof. Noting that \(a^n(a^{-1}; q)_n = (-1)^n q^{n(n-1)/2}\) for \(a = 0\), we see that

\[ (-q; q)_{\infty} J_{\frac{k-1}{2}, \frac{d}{2}}(0, 1, q^2) \]

\[ = (-q; q)_{\infty} (H_{\frac{k-1}{2}, \frac{d}{2}}(0, q^2, q^2) - 0 \cdot q^2 H_{\frac{k-1}{2}, \frac{d}{2} - 1}(0, q^2, q^2)) \]

\[ = (-q; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{nk^2+(k-d)n}q^{-n^2}}{(q^2; q^2)_n(q^2q^{2n}; q^2)_{\infty}} \]

\[ = 1 \cdot 1 \cdot 1 \cdot \cdots \]

Where using the Jacobi triple product we see

\[ \sum_{n=-\infty}^{\infty} (q^k)^{n^2} (-q^{-k-d})^n = \prod_{n=1}^{\infty} (1 - (q^k)^{2n})(1 - q^{-d}q^{k(2n-1)})(1 - q^{-k-d}q^{k(2n-1)}) = \]

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\[ = \prod_{n=1}^{\infty} (1 - q^{2kn})(1 - q^{2kn-d})(1 - q^{2k(n-1)+d}) \]
\[ = (q; q)_\infty \prod_{n \equiv 0, \pm d \pmod{2k}} (1 - q^n)^{-1}. \]

Now, by combining the two results we obtain the lemma. ■

**Proof of Theorem 4.3.** Denote by \( A_{k,d}(N) \) the number of partitions of the integer \( N \) into parts \( \not\equiv 0, \pm d \pmod{2k} \), and by \( B_{k,d}(N) \) the number of partitions of \( N \) of the form \( N = \sum_{j=1}^{M} a_j \) such that the conditions stated in Theorem 4.3. We wish to show that \( A_{k,d}(N) = B_{k,d}(N) \).

By (44) and (45) we have the following for \( 1 \leq d \leq k-1 \)
\[ J_{k-1/2,d}(0,0,q^2) = H_{k-1/2,d}(0,0,q^2) = 1 \]
and
\[ J_{k-1/2,d}(0,x^2,0) = H_{k-1/2,d}(0,0,0) = 1, \]
and therefore we have
\[ J_{k-1/2,d}(0,0,q^2) = J_{k-1/2,d}(0,x^2,0) = 1. \quad (46) \]

By definitions (44) and (45) we also have
\[ (-xq; q)_\infty J_{k-1/2,0}(0,x^2,q^2) = (-xq; q)_\infty H_{k-1/2,0}(0,x^2q^2,q^2) = 0 \quad (47) \]
and
\[ (-xq; q)_\infty J_{k-1/2,-1/2}(0,x^2,q^2) =
= (-xq; q)_\infty H_{k-1/2,1/2}(0,x^2q^2,q^2)
= (-xq; q)(-xq)^{-1}H_{k-1/2,1/2}(0,x^2q^2,q^2)
= -(xq)^{-1}(-xq; q)_\infty J_{k-1/2,1/2}(0,x^2,q^2). \quad (48) \]

For \( |x| \leq 1 \) and \( |q| < 1 \) we can express \((-xq; q)_\infty J_{k-1/2,d}(0,x^2,q^2)\) in the following way
\[ (-xq; q)_\infty J_{k-1/2,d}(0,x^2,q^2) = \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} a_{k,d}(M,N)x^M q^N. \quad (49) \]

Then by (47) we know that
\[ a_{k,0}(M,N) = 0. \quad (50) \]

By comparing the coefficients of \( x^M q^N \) in (48) we see that
\[ a_{k,-1}(M,N) = -a_{k,1}(M+1,N+1). \quad (51) \]

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The result in (46) yields
\[ a_{k,d}(M,N) = \begin{cases} 1 & \text{if } M = N = 0 \\
0 & \text{if either } M \leq 0 \text{ or } N \leq 0 \text{ and } (M,N) \neq (0,0). \end{cases} \] (52)

Using Lemma 4.3.3 we can find a forth condition on \( a_{k,d}(M,N) \) in the following way
\[
\sum_{M=0}^{\infty} \sum_{N=0}^{\infty} a_{k,d}(M,N)x^{M}q^{N} - \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} a_{k,d-2}(M,N)x^{M}q^{N} =
= (1 + xq)(xq)^{d-2} \left( \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} a_{k,k-d+1}(M,N)(xq)^{M}q^{N} + 0 \right)
= \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} a_{k,k-d+1}(M,N)x^{M+d-2}q^{N+d-2} + \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} a_{k,k-d+1}(M,N)x^{M+d-1}q^{N+d-1}.
\]

Comparing the coefficients of \( x^{M}q^{N} \) in he above we see that
\[ a_{k,d}(M,N) - a_{k,d-2}(M,N) = a_{k,k-d+1}(M-d+1, N-M) + a_{k,k-d+1}(M-d+2, N-M). \] (53)

Using mathematical induction it is easy to verify that the four conditions (50), (51), (52) and (53) uniquely determine the \( a_{k,d}(M,N) \) for \(-1 \leq d \leq k\).

Now let \( b_{k,d}(M,N) \) denote the number of partitions of \( N \) into \( M \) parts of the form \( N = \sum_{j=1}^{M} \alpha_{j} \) such that the conditions stated in the theorem are satisfied by the \( \alpha_{j} \). Furthermore set \( b_{k,0}(M,N) = 0 \) and \( b_{k,-1}(M,N) = -b_{k,1}(M+1, N+1) \). We wish to show that \( a_{k,d}(M,N) = b_{k,d}(M,N) \), i.e.
we wish to show that the \( b_{k,d}(M,N) \) satisfy (50), (51), (52) and (53) for \(-1 \leq d \leq k\).

By definition of \( b_{k,d}(M,N) \) we can immediately see that \( b_{k,d}(M,N) \) satisfies the first three conditions. We can also see that for \( M \geq N \) or \( d > M \) \( b_{k,d}(M,N) \) satisfies (53) by the third condition. For \( M < N \) and \( 2 \leq d \leq M \) we first write \( N \) in the following way
\[ N = \sum_{j=1}^{M} \alpha_{j} = \sum_{j=1}^{\infty} \beta_{j} \cdot j, \]

where \( \beta_{j} \geq 0 \) denotes the number of times the summand \( j \) appears in the partition. Obviously we have that \( \sum_{j=1}^{\infty} \beta_{j} = M \). In this case we note that \( b_{k,d}(M,N) - b_{k,d-2}(M,N) \) is the number of partitions of \( N \) into \( M \) parts of the form \( \sum_{j=1}^{M} \alpha_{j} = \sum_{j=1}^{\infty} \beta_{j} \cdot j \) where \( \beta_{1} = d - 1 \) or \( d - 2 \). If \( \beta_{1} = d - 1 \) we
see that \( \beta_2 \leq k - 1 - (d - 1) = k - d \) since \((d - 1) \cdot 1 + (k - d) \cdot 2 \equiv d - 1 \pmod{2}\). By subtracting 1 from every summand we obtain

\[
\sum_{j=1}^{\infty} \beta_j (j - 1) = \sum_{j=1}^{\infty} \beta_j \cdot j - \sum_{j=1}^{\infty} \beta_j = N - M,
\]

where the left-hand side also can be expressed as

\[
\sum_{j=1}^{\infty} \beta_j (j - 1) = \sum_{j=1}^{\infty} \beta_{j+1} \cdot j = \sum_{j=1}^{M-d+1} \tilde{\alpha}_j,
\]

with \( \tilde{\beta}_j = \beta_{j+1} \) and \( \tilde{\alpha}_j = \alpha_j - 1 \). Obviously we have that \( \sum_{j=1}^{\infty} \tilde{\beta}_j = M - d + 1 \). So we obtain a partition of \( N - M \) into \( M - d + 1 \) parts with \( \tilde{\alpha}_j \geq \tilde{\alpha}_{j+1} \), \( \tilde{\alpha}_j \geq \tilde{\alpha}_{j+k-1} + 2 \), \( \tilde{\beta}_1 = \beta_2 \leq k - d \) and if \( \tilde{\alpha}_j \leq \tilde{\alpha}_{j+k-2} + 1 \) we have \((\tilde{\alpha}_j + 1) \leq (\tilde{\alpha}_{j+k-2} + 1) + 1 \) which by previous assumption on \( \alpha_j \) implies that \((\tilde{\alpha}_j + 1) + \cdots + (\tilde{\alpha}_{j+k-2}) \equiv d - 1 \pmod{2} \) and thus \( \tilde{\alpha}_1 + \tilde{\alpha}_{j+1} + \cdots + \tilde{\alpha}_{j+k-2} \equiv k - d \pmod{2} \). Consequently our partition has been transformed into one enumerated by \( b_{k,d}(M-d+1, N-M) \). If \( \beta_1 = d - 2 \) it follows that \( \beta_2 \leq k - d \), since if it were \( k - d + 1 \) instead we would have \((d-2) \cdot 1 + (k-d) \cdot 2 \neq d - 1 \pmod{2} \). Repeating the above procedure we obtain a partition of \( N - M \) into \( M - d + 2 \) parts with \( \tilde{\alpha}_j \geq \tilde{\alpha}_{j+1} \), \( \tilde{\alpha}_j \geq \tilde{\alpha}_{j+k-1} + 2 \), if \( \tilde{\alpha}_j \leq \tilde{\alpha}_{j+k-2} + 1 \) we have \( \tilde{\alpha}_1 + \tilde{\alpha}_{j+1} + \cdots + \tilde{\alpha}_{j+k-2} \equiv k - d \pmod{2} \) and \( \tilde{\beta}_1 = \beta_2 \leq k - d \). So we end up with a partition enumerated by \( b_{k,k-d+1}(M-d+2, N-M) \). For the case \( M < N \) and \( d = 1 \) condition (53) follows by pretty much the same argument as above.

By uniqueness of \( a_{k,d}(M, N) \) it then follows that

\[
a_{k,d}(M, N) = b_{k,d}(M, N), \tag{54}
\]

for \(-1 \leq d \leq k\).

Setting \( x = 1 \), noting that \( B_{k,d}(N) = \sum_{M=0}^{\infty} b_{k,d}(M, N) \) and using (49), (54) and Lemma 4.3.4 we have

\[
\sum_{N=0}^{\infty} B_{k,d}(N) q^N = \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} b_{k,d}(M, N) q^N = \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} a_{k,d}(M, N) q^N = \left( -q; q \right)_{\infty} J_{\frac{k-1}{2}} (0, 1, q^2) = \prod_{n \neq 0, \pm d (\text{mod } 2k)} (1 - q^n)^{-1} = \sum_{N=0}^{\infty} A_{k,d}(N) q^N.
\]

By comparing the coefficients of \( q^N \) in the above the result follows. \( \blacksquare \)
4.4 Proof of Theorem 4

Proof of Theorem 4. For $i = 1$ and $k$, $d$ arbitrary the result follows by Theorem 4.1. By Theorem 4.2 and Theorem 4.2’ the result holds for $i = 0$, $k$ odd and $d$ arbitrary. Finally, for $i = 0$, $k$ even and $d$ arbitrary the result follows by Theorem 4.3.

As mentioned in the remark of Subsection 4.2 Andrews has also presented a proof to Gordon’s generalization of the Rogers-Ramanujan identities. We can see in [3] that this proof is based on the same idea as both Andrews’ proof of Theorem 4.2 and Bressoud’s proof of Theorem 4.3. Thus it should be possible to give a proof of Theorem 4 solely based on Andrews’ idea. We have however chosen not to do so, instead we have chosen to present a variety of different approaches to a similar problem.

In Gordon’s proof of Theorem 4.1 we see that it is possible to give a purely combinatorial proof to a generalization of The Rogers-Ramanujan identities. Andrews’ proof of Theorem 4.2 show us how a more analytic approach can be used to a similar problem. From Bressoud’s proof of Theroem 4.3 we see that it is possible to generalize Andrews’ idea in Subsection 4.2. Finally by Andrews’ proof of Gordon’s generalization we see that it is possible to even generalize Bressoud’s proof, i.e. to further generalize the idea in the proof of Theorem 4.2.
5 Some Fundamental Properties of Elliptic Functions

5.1 Theta Functions and Elliptic Functions

In a majority of proofs presented in the previous sections we use the Jacobi triple product, which states that for \(|q| < 1\) and \(x \neq 0\) we have

\[
s\sum_{n=-\infty}^{\infty} x^n q^n = \prod_{n=0}^{\infty} (1 - q^{2n+2})(1 + xq^{2n+1})(1 + x^{-1}q^{2n+1}).
\]

By replacing \(q\) with \(e^{\tau \pi i} (Im(\tau) > 0)\) and \(x\) with \(e^{2v \pi i} (v \text{ fixed})\) the left hand side in the Jacobi triple product becomes an absolutely convergent series,

\[
\sum_{n=-\infty}^{\infty} e^{2v \pi in} e^{\tau \pi in^2} = \sum_{n=-\infty}^{\infty} e^{(2vn+\tau n^2)\pi i}.
\]

(55)

Indeed, since \(Im(\tau) > 0\) we have \(|e^{\tau \pi i}| < 1\).

The expression in (55) is a trigonometric series in \(v\) and an example of a periodic function. In all of what follows it will be denoted by

\[
\theta(v, \tau) = \sum_{n=-\infty}^{\infty} e^{(2vn+\tau n^2)\pi i},
\]

and called the theta function. Obviously the following holds

\[
\theta(v + 1, \tau) = \theta(v, \tau).
\]

(56)

Since

\[
\theta(v, \tau) = 1 + \sum_{n=1}^{\infty} e^{(2vn+\tau n^2)\pi i} + \sum_{n=-\infty}^{-1} e^{(2vn+\tau n^2)\pi i}
\]

\[
= 1 + \sum_{n=1}^{\infty} e^{(2vn+\tau n^2)\pi i} + \sum_{n=1}^{\infty} e^{(-2vn+\tau n^2)\pi i}
\]

\[
= 1 + \sum_{n=1}^{\infty} e^{\tau n^2 \pi i} (e^{2vn \pi i} + e^{-2vn \pi i})
\]

\[
= 1 + \sum_{n=1}^{\infty} e^{\tau n^2 \pi i} 2 \cos(2vn \pi),
\]

we see that our \(\theta(v, \tau)\) is an entire function with respect to \(v\).
Moreover, since
\[
\theta(v + \tau, \tau) = \sum_{-\infty}^{\infty} e^{(2\tau \tau + 2\tau n + \tau n^2)\pi i}
\]
\[
= e^{-\pi i (\tau + 2\tau)} \sum_{-\infty}^{\infty} e^{(2\tau (n+1) + \tau (n+1)^2)\pi i}
\]
\[
= e^{-\pi i (\tau + 2\tau)} \sum_{-\infty}^{\infty} e^{(2\tau n + \tau n^2)\pi i},
\]
we have
\[
\theta(v + \tau, \tau) = e^{-\pi i (\tau + 2\tau)} \theta(v, \tau).
\] (57)

We will now use the above described theta function \( \theta(v, \tau) \) to construct an elliptic function \( \varphi \). By an elliptic function we mean a doubly periodic meromorphic function.

Set
\[
\varphi(v) = \frac{d^2}{dv^2} \ln \theta(v, \tau).
\]

By (56) and (57) we then have
\[
\varphi(v + 1) = \varphi(v),
\]
and
\[
\varphi(v + \tau) = \varphi(v).
\]

Since \( \theta(v, \tau) \) is an entire function the only singularities of \( \frac{d}{dv} \ln \theta(v, \tau) \) are simple poles. Now since the simple poles of \( \frac{d}{dv} \ln \theta(v, \tau) \) coincide with the zeros of \( \theta(v, \tau) \) the only singularities of \( \varphi(v) \) are poles of order two. Thus \( \varphi(v) = \frac{d^2}{dv^2} \ln \theta(v, \tau) \) is a doubly periodic function with poles of multiplicity two.

### 5.2 Liouville’s Theorems on Elliptic Functions

In this last subsection we will focus on proving four important theorems on elliptic functions due to Liouville. But before proceeding to the theorems and their proofs some preparatory work is needed.

Let \( \omega_1 \) and \( \omega_2 \) be two nonzero complex numbers such that \( \tau = \omega_1/\omega_2 \notin \mathbb{R} \) and \( Im(\tau) > 0 \). Furthermore let \( \varphi(v) \) be an elliptic function with periods \( 2\omega_1 \) and \( 2\omega_2 \). We now construct a period parallelogram \( P \) with vertices \( z_0, z_0 + 2\omega_1, z_0 + 2\omega_2 \) and \( z_0 + 2\omega_1 + 2\omega_2 \) by only including the vertex \( z_0 \), the two sides meeting at \( z_0 \) and the inside of the described parallelogram.

Two points \( z \) and \( z' \) are denoted by \( z \equiv z' (mod \ (2\omega_1, 2\omega_2)) \) and said to be congruent modulo the periods \( 2\omega_1 \) and \( 2\omega_2 \) if for \( n_1, n_2 \in \mathbb{Z} \) we have \( z - z' = n_1 2\omega_1 + n_2 2\omega_2 \). Since the vertices \( z_0 + 2\omega_1, z_0 + 2\omega_2 \) and \( z_0 + 2\omega_1 + 2\omega_2 \)
are not included in our period parallelogram \( P \) we note that it cannot contain two congruent points.

Now let \( r_1 \) and \( r_2 \) be two real numbers such that

\[
z - z_0 = r_1 2\omega_1 + r_2 2\omega_2,
\]

and set \( r_1 = n_1 + s_1 \) and \( r_2 = n_2 + s_2 \), where \( 0 \leq s_1, s_2 < 1 \) and \( n_1, n_2 \) are integers. We then have

\[
z - (z_0 + s_1 2\omega_1 + s_2 2\omega_2) = n_1 2\omega_1 + n_2 2\omega_2.
\]

Consequently we see that every point \( z \in \mathbb{C} \) is congruent to a unique point in the described period parallelogram \( P \). Corresponding to all systems of mutually congruent points is a system of period parallelograms that covers the whole complex plane. So when studying an elliptic function on \( \mathbb{C} \) it is enough to study it in a period parallelogram. Conveniently, because the vertex \( z_0 \) is arbitrarily chosen and because every meromorphic function takes each value finitely many times in a finite region we can construct the period parallelogram \( P \) so that our elliptic function \( \varphi \) does not attain certain values on the boundary of \( P \).

The following four results are all more or less almost immediate consequences of Cauchy’s theorem.

**Proposition 5.1** Let \( \varphi \) be an elliptic function and let \( c \) be an arbitrary complex number. Then the number of zeros, counted with multiplicity, of \( \varphi(z) - c \) in a period parallelogram \( P \) equals the number of poles of \( \varphi(z) \) in \( P \).

**Proof.** Let \( k \) be the number of zeros of \( \varphi(z) - c \) in \( P \) with vertices \( z_0, z_0 + 2\omega_1, z_0 + 2\omega_2 \) and \( z_0 + 2\omega_1 + 2\omega_2 \). Furthermore, let \( l \) be the number of poles of \( \varphi(z) - c \) in the same \( P \). Then by Cauchy’s theorem we know that

\[
\int_{\partial P} \frac{\varphi'(z)}{\varphi(z) - c} \, dz = 2\pi i (k - l),
\]

where

\[
\int_{\partial P} \frac{\varphi'(z)}{\varphi(z) - c} \, dz = \int_{z_0}^{z_0 + 2\omega_1 + 2\omega_2} \frac{\varphi'(z)}{\varphi(z) - c} \, dz + \int_{z_0 + 2\omega_1}^{z_0 + 2\omega_1 + 2\omega_2} \frac{\varphi'(z)}{\varphi(z) - c} \, dz + \int_{z_0 + 2\omega_2}^{z_0 + 2\omega_1 + 2\omega_2} \frac{\varphi'(z)}{\varphi(z) - c} \, dz + \int_{z_0 + 2\omega_1}^{z_0} \frac{\varphi'(z)}{\varphi(z) - c} \, dz.
\]

Substituting \( z \) with \( z + 2\omega_1 \) in the second integral we obtain

\[
\int_{z_0 + 2\omega_1}^{z_0 + 2\omega_1 + 2\omega_2} \frac{\varphi'(z)}{\varphi(z) - c} \, dz = - \int_{z_0 + 2\omega_2}^{z_0} \frac{\varphi'(z)}{\varphi(z) - c} \, dz,
\]

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due to periodicity of $\varphi$. This now cancels the forth integral on the right hand side above. Similarly, replacing $z$ with $z + 2\omega_2$ in the third integral we see that

$$\int_{z_0+2\omega_2}^{z_0+2\omega_1+2\omega_2} \frac{\varphi'(z)}{\varphi(z) - c} \, dz = -\int_{z_0}^{z_0+2\omega_1} \frac{\varphi'(z)}{\varphi(z) - c} \, dz,$$

which cancels the first integral. Thus we have

$$\int_{\partial P} \frac{\varphi'(z)}{\varphi(z) - c} \, dz = 0.$$ 

It then follows that $2\pi i (k - l) = 0$. Therefore the number of zeros of $\varphi(z) - c$ is equal to the number of poles of $\varphi(z) - c$. Since any pole of $\varphi(z) - c$ is also a pole of $\varphi(z)$, and vice versa, the statement follows.

Proposition 5.2 There does not exist a nonconstant elliptic function that is regular in a period parallelogram.

Proof. Let $\varphi$ be such a function. The number of poles of $\varphi$ is then zero. By Proposition 5.1 we then know that the number of zeros to $\varphi(z) - c$ is zero for any arbitrary $c$, which is an absurdity. ■

Proposition 5.3 The number of poles of an elliptic function in a period parallelogram must be at least two.

Proof. Let $\varphi$ be an elliptic function in a period parallelogram $P$ with periods $2\omega_1$ and $2\omega_2$. By Cauchy’s theorem we know that

$$\int_{\partial P} \varphi(z) \, dz = 2\pi i \sum_{j=1}^{m} \text{Res}[\varphi(z), z_j],$$

where $z_1, \ldots, z_m$ are the poles of $\varphi$ inside $P$. By the same variable substitution as in the proof of Proposition 5.1 we then see

$$\int_{\partial P} \varphi(z) \, dz = \int_{z_0}^{z_0+2\omega_1} \varphi(z) \, dz + \int_{z_0+2\omega_1+2\omega_2}^{z_0+2\omega_1} \varphi(z) \, dz + \int_{z_0+2\omega_2}^{z_0+2\omega_1+2\omega_2} \varphi(z) \, dz$$

$$+ \int_{z_0+2\omega_1+2\omega_2}^{z_0+2\omega_2} \varphi(z) \, dz + \int_{z_0}^{z_0+2\omega_2} \varphi(z) \, dz = 0.$$ 

Thus the sum of the residues of elliptic functions with respect to all poles inside $P$ is equal to zero. Now, since a function with a simple pole needs to have a nonzero residue with respect to that pole we know by the above stated that it cannot be elliptic. ■

Proposition 5.4 Let $\varphi$ be an elliptic function with periods $2\omega_1$, $2\omega_2$ and let $c$ be an arbitrary number. Denote by $c_1, \ldots, c_k$ the zeros of $\varphi(z) - c$ in
a period parallelogram $P$, and by $b_1, \ldots, b_k$ the poles of \( \varphi(z) \) in the same period parallelogram $P$. Then,

\[
\sum_{j=1}^{k} c_j \equiv \sum_{j=1}^{k} b_j \pmod{2\omega_1, 2\omega_2}.
\]

**Remark.** Note that in the above stated proposition, Proposition 5.4, the number of zeros of \( \varphi(z) - c \) equals the number of poles of \( \varphi(z) \). This result follows by Proposition 5.1. ▲

**Proof.** By Cauchy’s theorem we know

\[
\int_{\partial P} z \frac{\varphi'(z)}{\varphi(z) - c} \, dz = 2\pi i \left( \sum_{j=1}^{k} c_j - \sum_{j=1}^{k} b_j \right).
\]

Let $P$ have vertices in $z_0$, $z_0 + 2\omega_1$, $z_0 + 2\omega_2$ and $z_0 + 2\omega_1 + 2\omega_2$. As before we have the following

\[
\int_{\partial P} z \frac{\varphi'(z)}{\varphi(z) - c} \, dz = \int_{z_0}^{z_0 + 2\omega_1} z \frac{\varphi'(z)}{\varphi(z) - c} \, dz + \int_{z_0 + 2\omega_1}^{z_0 + 2\omega_1 + 2\omega_2} z \frac{\varphi'(z)}{\varphi(z) - c} \, dz + \int_{z_0 + 2\omega_1 + 2\omega_2}^{z_0 + 2\omega_2} z \frac{\varphi'(z)}{\varphi(z) - c} \, dz + \int_{z_0 + 2\omega_2}^{z_0} z \frac{\varphi'(z)}{\varphi(z) - c} \, dz.
\]

Again, using the same variable substitution as in the proof of Proposition 5.1 we obtain

\[
\int_{z_0}^{z_0 + 2\omega_1} z \frac{\varphi'(z)}{\varphi(z) - c} \, dz + \int_{z_0 + 2\omega_1}^{z_0 + 2\omega_2} z \frac{\varphi'(z)}{\varphi(z) - c} \, dz = \int_{z_0}^{z_0 + 2\omega_1} z \frac{\varphi'(z)}{\varphi(z) - c} \, dz - \int_{z_0}^{z_0 + 2\omega_1} (z + 2\omega_2) \frac{\varphi'(z)}{\varphi(z) - c} \, dz = 2\omega_2 \int_{z_0}^{z_0 + 2\omega_1} \frac{\varphi'(z)}{\varphi(z) - c} \, dz = 2\omega_2 \left[ \ln(\varphi(z) - c) \right]_{z_0}^{z_0 + 2\omega_1} = 4\pi i \omega_2 n_2,
\]

where $n_2 \in \mathbb{N}$. Similarly

\[
\int_{z_0 + 2\omega_1}^{z_0 + 2\omega_2} z \frac{\varphi'(z)}{\varphi(z) - c} \, dz + \int_{z_0 + 2\omega_2}^{z_0} z \frac{\varphi'(z)}{\varphi(z) - c} \, dz = \int_{z_0}^{z_0 + 2\omega_2} (z + 2\omega_1) \frac{\varphi'(z)}{\varphi(z) - c} \, dz - \int_{z_0}^{z_0 + 2\omega_2} z \frac{\varphi'(z)}{\varphi(z) - c} \, dz = 2\omega_1 \int_{z_0}^{z_0 + 2\omega_1} \frac{\varphi'(z)}{\varphi(z) - c} \, dz = \omega_1 \left[ \ln(\varphi(z) - c) \right]_{z_0}^{z_0 + 2\omega_2} = 4\pi i \omega_1 n_1,
\]

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where \( n_1 \in \mathbb{N} \). Thus

\[
\sum_{j=1}^{k} c_j - \sum_{j=1}^{k} b_j = n_1 2\omega_1 + n_2 2\omega_2,
\]

and we have the result. ■
References


