Lagrange models for asymmetric random waves

Georg Lindgren
Mathematical Statistics, Lund University
georg@maths.lth.se

Abstract

The Lagrange model for ocean waves is a hydrodynamically motivated extension of the Gaussian model – used in ocean engineering since the mid 50’s. It consists of a Gaussian space and time dependent process for the vertical height of water particles, and an accompanying horizontal displacement process for the horizontal position of the particles. It is shown how by suitable choice of structure, one can obtain waves with realistic crest-trough and front-back asymmetry. It is also shown how one can calculate the exact statistical distribution of many wave characteristics.

1. Lagrange models

A Lagrange wave, a stochastic version of a Miche wave, describes the vertical and horizontal movements of individual water particle as functions of time \( t \) and original location \( u \). In the stochastic Lagrange model the vertical and horizontal displacements are correlated random processes. The vertical process, \( W(t, u) \), describes elevation above the still water level and is taken as a Gaussian process with mean zero, and so is the horizontal location, \( X(t, u) \).

The Lagrange model is defined by the covariance functions \( r_{XX}(t, u) \), \( r_{XW}(t, u) \), \( r_{WW}(t, u) \).

The height of the surface at location \( X(t, u) \) is equal to \( W(t, u) \).

The Lagrange model is a linear filtration of \( x_{\infty}(t, u) \), leading to the cross-covariance function

\[
r_{XX}(t, u) = \int_0^\infty \cos(\omega t - \theta_0(\omega)) \rho(\omega) S(\omega) d\omega.
\]

A physically motivated relation is obtained by letting the horizontal acceleration be, e.g.,

\[
\frac{\partial^2 X(t, u)}{\partial t^2} + \alpha \frac{\partial X(t, u)}{\partial t} = \rho(\omega) \rho^*(-\omega) W(t, u),
\]

with \( \alpha > 0 \). The response function will then be

\[
H(\omega) = \frac{\cosh(i\omega)}{\sinh(i\omega)} \frac{\alpha}{\rho^2(\omega) + \alpha^2}.
\]

2. Wave characteristics

The wave is the curve \( u \rightarrow (X(t, u), W(t, u)) \), keeping time \( t = t_0 \) fixed; defined implicitly through the relation \( L(t_0, X(t_0, u)) = W(t_0, u) \). The time wave is obtained as measurements of the free water level \( L(t, x_0) \) at a fixed location in space with co-ordinate \( x_0 \); viz. as the curve \( t \rightarrow W(t, X(t, x_0)) \). Slopes are defined, for time and space, as

\[
L_t(t, X(t, u)) = W_t(t, u) - W(t, u) X_t(t, u)
\]

and

\[
L_{xx}(t, x_0) = W_{xx}(t, x_0) - W(t, x_0) X_{xx}(t, x_0).
\]

(SS) Slope in space at level crossings in space.

The distribution of the space slope \( L_{xx}(t, x_0) \), observed at the up- or downcrossings of a fixed level \( \nu \) by the space wave \( L(t, x_0) \)

- at any level \( \nu \) and time \( t \), if there is a particle with reference coordinate \( u_0 \) such that \( W(t_0, u_0) = \nu \) and \( X(t_0, u_0) = x_0 \).

(TT) Slope in time at level crossings in space.

The distribution of the time slope \( L_t(t, x_0) \) observed at the up- or downcrossings of a fixed level \( \nu \) by the time wave \( L(t, x_0) \)

- at any level \( \nu \) and time \( t \), if there is a particle with reference coordinate \( u_0 \) such that \( W(t_0, u_0) = \nu \), and \( X(t_0, u_0) = x_0 \); evaluated at \( (x, t) \), and cut off \( p \).

3. Asymmetry in space and time

The distribution of wave slope at \( \nu \)-level upcrossing is the distribution of the defining ratio, conditioned on that \( u \) is a \( \nu \)-upcrossing in the vertical Gaussian process \( W(t, u) \), with Rayleigh distributed slope. Let \( R \) and \( U \) be independent standard Rayleigh and normal variables. The notation \( X \leftrightarrow Y \) means “equal in distribution”.

Theorem 1 The slope of the Lagrange space wave at \( \nu \)-upcrossing has the representation \( L_t(t, x_0) \)

\[
\int \frac{dU}{U} R\sqrt{\frac{W_s}{W_s + U^2}} - \frac{R^2}{W_s + U^2}.
\]

Front-back asymmetry depends on the covariance function

\[
r_{Wx}(u) = \int_0^\infty \int \frac{\rho(\omega) \rho^*(\omega)}{-\omega} \omega S(\omega) d\omega.
\]

between the spatial derivatives of the vertical and horizontal processes.

Figure 1 illustrates the asymmetry in case (SS).

Figure 1: CDF for slopes at upcrossings (solid) and negative slopes at downcrossings (dashed) of levels -1, 0, 1, for linked Lagrange space waves with \( \alpha = 0.4 \). Smallest slope at level -1, largest at level +1.

Figure 2: Cumulative distribution functions for time wave slopes (absolute values) at time wave crossings of different levels. Slope CDF at upcrossings (solid lines) and at downcrossings (dashed-dotted lines). Levels \( \nu = [-1, 0, 1, 2, 3] \times \sigma \), \( \sigma = H_0 \). Largest absolute values correspond to highest level.

References